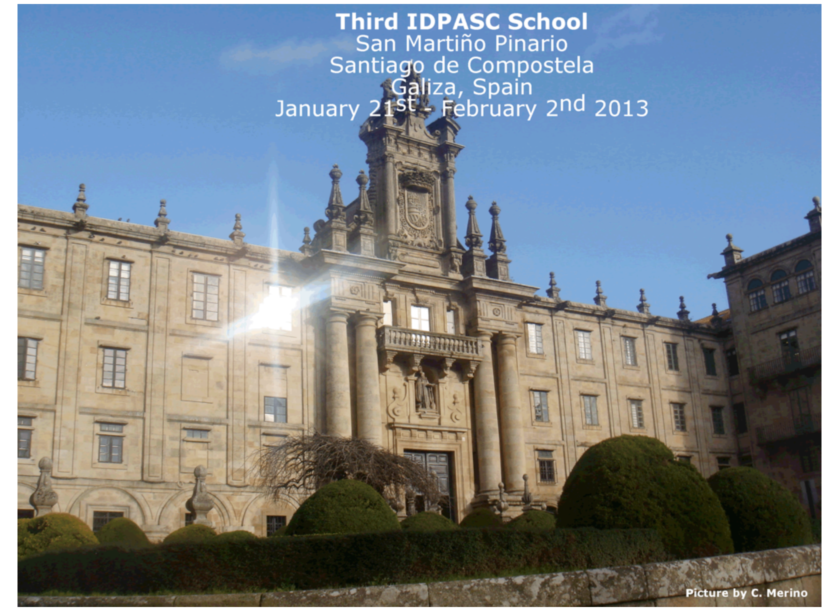

Introduction to AdS/CFT

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Thermodynamics

canonical partition function → $Z = \mathrm{Tr}\, e^{-\frac{H}{T}}$

Thermal average → $\langle \mathcal{O} \rangle_T = \dfrac{\mathrm{Tr}\big[\mathcal{O}\, e^{-\frac{H}{T}}\big]}{Z}$ →

$$\langle \mathcal{O} \rangle_T \sim \int [D\psi] \langle \psi(x), t\, | \mathcal{O}\, e^{-\frac{H}{T}} | \psi(x), t \rangle \;\longrightarrow\; \langle \mathcal{O} \rangle_T \sim \int [D\psi]\, \langle \psi(x), t\, | \mathcal{O} | \psi(x), t + \frac{i}{T} \rangle$$

→ imaginary time evolution+ (anti)periodic b. c.

Euclidean time identification → $t_E \equiv t_E + \dfrac{1}{T}$ → Thermal circle

Black hole thermodynamics

BH geometry →

$$ds^2 = g(r)\,\big[\, f(r)\, dt_E^2 \,+\, d\vec{x}^{\,2} \,\big] \,+\, \frac{1}{h(r)}\, dr^2$$

$f(r),\; h(r) \rightarrow$ first-order zero at $r = r_0$ $\qquad g(r_0) \neq 0$

$r = r_0$ is the horizon of the black hole near-horizon → $f(r) \approx f'(r_0)\,(r-r_0)$ $h(r) \approx h'(r_0)\,(r-r_0)$ $g(r) = g(r_0)$

$$\rightarrow \quad ds^2 \approx g(r_0)\left[f'(r_0)\,(r-r_0)dt_E^2 + d\vec{x}^{\,2}\right] + \frac{1}{h'(r_0)}\,\frac{dr^2}{r-r_0}$$

Change variables

$$\frac{1}{h'(r_0)}\,\frac{dr^2}{r-r_0} = d\rho^2 \quad \rightarrow \quad \rho = 2\sqrt{\frac{r-r_0}{h'(r_0)}}$$

$$g(r_0)\,f'(r_0)\,(r-r_0)dt_E^2 = \rho^2\,d\theta^2 \quad \rightarrow \quad \theta = \frac{1}{2}\sqrt{g(r_0)\,f'(r_0)\,h'(r_0)}\;t_E$$

(t_E, r) part of the metric $\rightarrow d\rho^2 + \rho^2 d\theta^2$

↓

Regular at $\rho = 0$ (the horizon) if $\theta \equiv \theta + 2\pi$

→ $t_E \equiv t_E + \frac{1}{T}$ →

non-zero temperature →

$$\frac{1}{T} = \frac{4\pi}{\sqrt{g(r_0)\,f'(r_0)\,h'(r_0)}}$$

Hawking temperature

Application to the Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$$

It corresponds to $\quad g(r) = 1 \qquad f(r) = h(r) = 1 - \dfrac{2GM}{r}$

horizon → $r = r_0 = 2GM$ → $f'(r_0) = h'(r_0) = \dfrac{2GM}{r_0^2}$ → $T = \dfrac{1}{8\pi GM}$

Black hole entropy

→ Identify M with the internal energy E

→ Use the first law of thermodynamics $dE = TdS$

$dM = TdS = \dfrac{1}{8\pi GM} dS$ → integrate → $S = 4\pi\, G\, M^2$

area of the horizon → $A_H = 4\pi\, r_0^2 \; = \; 16\, G^2\, M^2$

$$S = \frac{A_H}{4G}$$

Bekenstein-Hawking entropy formula

AdS black hole

Euclidean metric → $$ds^2 = \frac{L^2}{z^2}\left[f(z)\, dt_E^2 + d\vec{x}^{\,2} + \frac{dz^2}{f(z)} \right]$$ $$f(z) = 1 - \frac{z^d}{z_0^d}$$ blackening factor

$$g = \frac{L^2}{z^2} \qquad h = \frac{z^2}{L^2} f \quad \rightarrow \quad f'(z_0) = -\frac{d}{z_0} \qquad h'(z_0) = -\frac{d z_0}{L^2} \qquad g(z_0)\, f'(z_0)\, h'(z_0) = \frac{d^2}{z_0^2}$$

Hawking temperature → $$T = \frac{d}{4\pi z_0}$$

Finite temperature AdS/CFT correspondence

The AdS BH is dual to $\mathcal{N} = 4$ SYM at $T \neq 0$

Hawking temperature $\equiv$ temperature of the dual theory

area of the horizon → $$A_H = \left(\frac{L}{z_0}\right)^{d-1} V_{d-1} \quad \rightarrow \quad A_H = \left(\frac{4\pi}{d}\right)^{d-1} L^{d-1}\, T^{d-1} V_{d-1}$$

entropy → $$S = \frac{A_H}{4G_{d+1}} = \frac{1}{4G_{d+1}} \left(\frac{4\pi}{d}\right)^{d-1} L^{d-1}\, T^{d-1}\, V_{d-1}$$

entropy density $\rightarrow$ $s = \frac{S}{V_{d-1}}$ $\rightarrow$

$$s = \left(\frac{4\pi}{d}\right)^{d-1} c_{QFT}\, T^{d-1}$$

$$c_{QFT} = \frac{1}{4}\left(\frac{L}{l_P}\right)^{d-1}$$

$\mathcal{N} = 4$ SYM $\rightarrow$ $d = 4$ $\quad c_{SYM} = \frac{N^2}{2\pi}$ $\rightarrow$ $s_{SYM} = \frac{\pi^2}{2} N^2 T^3$

strong coupling

Comparison with field theory

Partition function of relativistic free particles

$$\log Z = \mp V_3 \int \frac{d^3p}{(2\pi)^3}\, \log\left(1 \mp e^{-\frac{\omega(p)}{T}}\right) \qquad \omega(p) = \sqrt{\vec{p}^2 + m^2}.$$

$- \longrightarrow$ bosons $\qquad + \longrightarrow$ fermions

Massless particles →

$$\frac{\log Z}{V_3} = \mp \int_0^\infty \frac{dp}{2\pi^2}\, p^2 \log\left(1 \mp e^{-\frac{p}{T}}\right) \quad \longrightarrow \quad \frac{\log Z}{V_3} = \frac{T^3}{6\pi^2}\int_0^\infty dx\, \frac{x^3}{e^x \mp 1}$$

Using →

$$\int_0^\infty dx\, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \qquad \int_0^\infty dx\, \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120}$$

$$\boxed{\frac{\log Z}{V_3} = \frac{\pi^2}{90} T^3}$$

bosons

$$\boxed{\frac{\log Z}{V_3} = \frac{7\pi^2}{720} T^3}$$

fermions

Entropy density →

$$s = \frac{\partial}{\partial T}\left[T\, \frac{\log Z}{V_3}\right] = 4\, \frac{\log Z}{V_3}$$

$$\boxed{s_{boson} = \frac{2\pi^2}{45} T^3} \qquad \boxed{s_{fermion} = \frac{7\pi^2}{180} T^3}$$

In $\mathcal{N}=4$ SYM

bosons $\longrightarrow$ $\left[\,2(\text{gauge field}) + 6(\text{scalar field})\,\right] N^2 \;=\; 8N^2$

fermions $\longrightarrow$ $\left[\,2\times 4(\text{Weyl spinors})\,\right] N^2 \;=\; 8N^2$

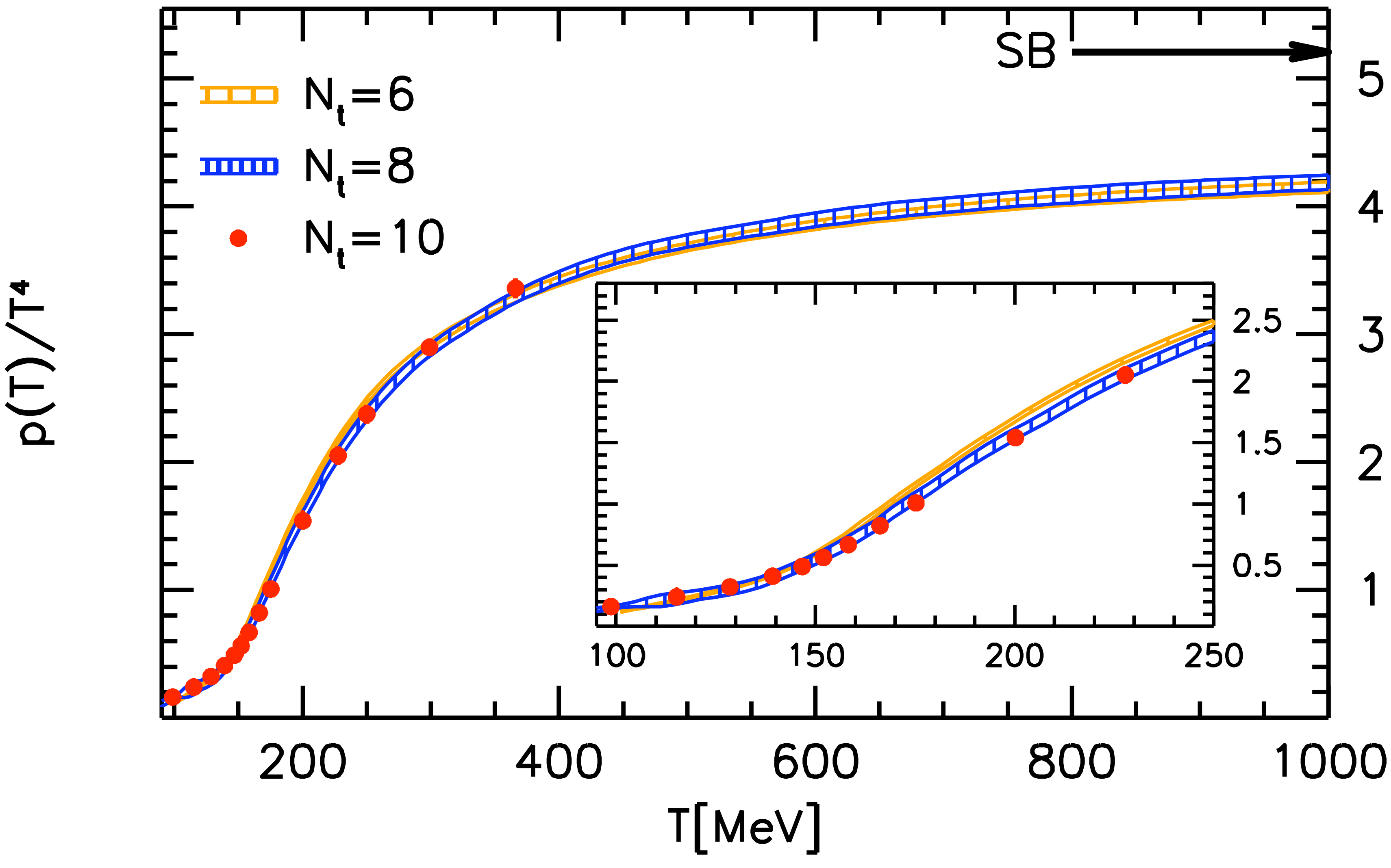
$$s_{\mathcal{N}=4\; free\; gas} = 8\,N^2 \left[\frac{2\pi^2}{45} + \frac{7\pi^2}{180}\right] T^3 \;=\; \frac{2\pi^2}{3}\,N^2\,T^3$$

$$s_{black\; hole} \;=\; \frac{3}{4}\, s_{\mathcal{N}=4\; free\; gas}$$

$\Downarrow$

$$p_{black\; hole} = \frac{3}{4}\, p_{\mathcal{N}=4\; free\; gas}$$

Numerical values from lattice QCD



SB=Stefan-Boltzman=free gas

Linear response in real time

Perturb the QFT by a source in Minkowski signature

$$S = S_0 + \int d^d x \, \mathcal{O}(x) \, \varphi(x)$$

one-point function ⟶ $\langle \mathcal{O}(x) \rangle_\varphi = -\int G_R(x-y) \varphi(y) \, dy$

$$i \, G_R(x-y) \equiv \theta(x^0 - y^0) \, \langle [\mathcal{O}(x), \mathcal{O}(y)] \rangle$$

⟶ Retarded Green's function ⟶ causality

Momentum space ⟶ $\langle \mathcal{O}(\omega, \vec{k}) \rangle_\varphi = -G_R(\omega, \vec{k}) \, \varphi(\omega, \vec{k})$

Long wavelength limit ⟶ Response to a time varying source

$$\langle \mathcal{O} \rangle_\varphi \approx -\chi \, \partial_t \, \varphi$$

⟶ $\chi \rightarrow \text{transport coefficient}$

Fourier transform in time ⟶ $\langle \mathcal{O} \rangle_\varphi \approx i \, \omega \, \chi \, \varphi(\omega)$

Example:

Take $\mathcal{O} = j^\mu \rightarrow$ conserved current $\Longrightarrow$ $\varphi = A^\mu \rightarrow$ gauge field

$\partial\varphi \sim \vec{E} \rightarrow$ electric field $\Longrightarrow$ $\chi = \sigma \rightarrow$ conductivity

$$\vec{J} = \sigma \vec{E}$$

Ohm's law

In general

Long wavelength limit $\rightarrow$ take $\vec{k} = 0$ and $\omega \rightarrow 0$

$$\langle \mathcal{O} \rangle_\varphi = -G_R(\omega, \vec{k} = 0)\, \varphi(\omega) \Longrightarrow G_R(\omega, \vec{k} = 0) = -i\,\omega\,\chi \qquad (\omega \rightarrow 0)$$

$$\chi = -\lim_{\omega \to 0} \lim_{\vec{k} \to 0} \frac{1}{\omega} \operatorname{Im} G_R(\omega, \vec{k})$$

Kubo formula

We will apply this formula to compute the shear viscosity

Transport coefficient of a massless scalar field

Consider a $(d+1)$-dimensional metric $\longrightarrow$ $ds^2 = g_{tt}\, dt^2 + g_{zz} dz^2 + g_{xx}\delta_{ij}\, dx^i dx^j$

$z_0 \rightarrow$ horizon of the black hole $\longrightarrow$

$$g_{tt} = -c_0(z_0 - z)$$

$$g_{zz} = \frac{c_z}{z_0 - z}$$

$z \approx z_0$

Action

$$S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,\frac{\partial_M\phi\,\partial^M\phi}{q(z)}$$

$q(z) \rightarrow$ effective coupling of the mode

eom $\longrightarrow$ $$\partial_M\left(\frac{\sqrt{-g}}{q}\,g^{MN}\,\partial_M\phi\right) = 0$$

canonical momentum $\longrightarrow$ $$\Pi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_z\phi)} = -\frac{\sqrt{-g}}{q}\,g^{zz}\,\partial_z\phi$$ $\longrightarrow$

$$\partial_z\,\Pi = \frac{\sqrt{-g}}{q}\left(\frac{\partial_t^2\phi}{g_{tt}} + \frac{\partial_i^2\phi}{g_{xx}}\right)$$

Massless field $\longrightarrow$ $\Delta = d$

$\longrightarrow$ The on-shell action is finite and $S_{ct} = 0$

$$\longrightarrow \quad \boxed{\langle \mathcal{O}(x) \rangle_{\varphi} = \lim_{z \to 0} \Pi(z, x)}$$

Momentum space

$$\phi(z, t, \vec{x}) = \int \frac{d\omega \, d^{d-1}k}{(2\pi)^d} \, e^{i(\vec{k}\cdot x - \omega t)} \, \phi(z, \omega, \vec{k}) \qquad \Pi(z, t, \vec{x}) = \int \frac{d\omega \, d^{d-1}k}{(2\pi)^d} \, e^{i(\vec{k}\cdot x - \omega t)} \, \Pi(z, \omega, \vec{k})$$

$$\boxed{G_R(k_\mu) = \lim_{z \to 0} \frac{\Pi(z, k_\mu)}{\phi(z, k_\mu)}}$$

To get the retarded Green's function we need to impose causal b. c.

Transport coefficient

$$\boxed{\chi = -\lim_{k_\mu \to 0} \lim_{z \to 0} \mathrm{Im} \left[\frac{\Pi(z, k_\mu)}{\omega \, \phi(z, k_\mu)} \right] = -\lim_{k_\mu \to 0} \lim_{z \to 0} \frac{\Pi(z, k_\mu)}{i\omega \, \phi(z, k_\mu)}}$$

Claim:

$$\text{Evaluating } \Pi/\omega\,\phi \text{ at the boundary} \equiv \text{evaluate it at the horizon}$$

Proof:

Consider the equation → $\partial_z \left[\, A(z)\, \partial_z \phi \,\right] \;=\; B(z)\, \phi(z)$

In hamiltonian form → $\Pi(z) \;\equiv A(z)\, \partial_z \phi(z) \qquad \partial_z\, \Pi(z) = B(z)\, \phi(z)$

Equivalent to the Riccati equation → $$\partial_z \left(\frac{\Pi(z)}{\phi(z)} \right) = B(z) - \frac{1}{A(z)} \left(\frac{\Pi(z)}{\phi(z)} \right)^2$$

In our case $\quad A(z) = -\frac{\sqrt{-g}}{q}\, g^{zz} \qquad B(z) = \frac{\sqrt{-g}}{q} \left[-\frac{\omega^2}{g_{tt}} - \frac{\vec{k}^{\,2}}{g_{xx}} \right]$

$$\partial_z \left[\frac{\Pi}{\omega\phi} \right] = \omega \left[\frac{q g_{zz}}{\sqrt{-g}} \left(\frac{\Pi}{\omega\phi} \right)^2 - \frac{\sqrt{-g}}{q\, g_{tt}} \left(1 + \frac{g_{tt}}{g_{xx}} \frac{\vec{k}^{\,2}}{\omega^2} \right) \right]$$

The rhs of this equation vanishes when the ordered limit $\lim_{\omega\to 0} \lim_{\vec{k}\to 0}$ is taken

$$\chi = -\lim_{k_\mu \to 0} \lim_{z \to z_0} \frac{\Pi(z, k_\mu)}{i\omega\, \phi(z, k_\mu)}$$

Near $z = z_0$ ➡ $\Pi \approx -\frac{1}{c_z} \frac{\sqrt{-g(z_0)}}{q(z_0)} (z_0 - z)\, \partial_z\, \phi$

eom ➡ $\partial_z \Big[(z_0 - z)\, \partial_z\, \phi(z, k^\mu) \Big] + c_z \Big[\frac{\omega^2}{c_0 (z_0 - z)} - \frac{k^2}{g_{xx}(z_0)} \Big]\, \phi(z, k^\mu) \;= 0$

solve eom as $\phi = (z_0 - z)^\beta$ ➡ $\beta = \pm i \sqrt{\frac{c_z}{c_0}}\, \omega$

Two solutions : $\phi_\pm \sim (z_0 - z)^{\pm i \sqrt{\frac{c_z}{c_0}}\, \omega}$

$\phi_+ \to$ outgoing waves at the horizon $\qquad \phi_- \to$ infalling waves at the horizon

causality ➡ infalling wave ➡ retarded Green's function

$\partial_z \phi_- = \sqrt{\frac{g_{zz}(z_0)}{-g_{tt}(z_0)}}\; i\omega\, \phi_-$ ➡ $\left.\frac{\Pi}{i\omega\, \phi}\right|_{z_0} = -\frac{1}{q(z_0)} \left.\sqrt{\frac{g}{g_{zz}\, g_{tt}}}\right|_{z_0}$

Transport coefficient → $$\chi = \frac{1}{q(z_0)} \sqrt{\frac{g}{g_{zz}\, g_{tt}}}\Bigg|_{z_0}$$

Equivalently

$$\chi = \frac{1}{q(z_0)} \frac{A_H}{V}$$ $A_H \rightarrow$ area of the horizon

Compare with the Bekenstein-Hawking formula for the entropy density

$$s = \frac{1}{4G_N} \frac{A_H}{V}$$

ratio → $$\frac{\chi}{s} = \frac{4G_N}{q(z_0)}$$

Relativistic hydrodynamics

Effective theory of continuous systems at large distances and time scales

Basic object $\to$ energy-momentum tensor $T^{\mu\nu}$ $\Rightarrow$ $\partial_\mu T^{\mu\nu} = 0$

Local thermal equilibrium $\Rightarrow$ state determined by local temperature $T(x)$ and fluid velocity $u^\mu(x)$

$$u_\mu u^\mu = -1$$

Constitutive relations $\Rightarrow$

$$T^{\mu\nu} = \underbrace{(\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu}}_{\text{ideal fluid}} - \underbrace{\sigma^{\mu\nu}}_{\text{dissipative}}$$

$\epsilon \to$ energy density $\quad$ $p \to$ pressure

Local rest frame $\to u^\mu = (1,0,0,0)$ and $\sigma^{00} = \sigma^{0i} = 0$ $\Rightarrow$

$$\sigma_{ij} = \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\,\delta_{ij}\,\partial_k u_k\right) + \zeta\,\delta_{ij}\partial_k u_k$$

$\eta \to$ shear viscosity $\quad$ $\zeta \to$ bulk viscosity

Covariant form of $\sigma^{\mu\nu}$ in a curved space

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + \left(\zeta - \frac{2}{3}\eta\right) g_{\alpha\beta} \nabla \cdot u \right]$$

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\mu_{\alpha\beta} u_\mu \qquad \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \right]$$

$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ ➡ Projector onto the directions perpendicular to u^μ

Linear response formulation ➡ $\sigma^{\mu\nu} \to$ one-point function of $T^{\mu\nu}$

Metric perturbation $g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$ ➡

$$\sigma^{\mu\nu}(x) = \int G_R^{\mu\nu,\alpha\beta}(x-y)\, h_{\alpha\beta}(y) dy$$

$$i\, G_R^{\mu\nu,\alpha\beta}(x-y) = \theta(x^0 - y^0) \langle [T^{\mu\nu}(x), T^{\alpha\beta}(y)] \rangle$$

Particular metric perturbation

$$g_{00}(t,\vec{x}) = -1 \qquad g_{0i}(t,\vec{x}) = 0 \qquad g_{ij}(t,\vec{x}) = \delta_{ij} + h_{ij}(t)$$

$$h_{ij} << 1 \text{ and such that is traceless } (h_{ii} = 0)$$

$$\Downarrow$$

$$\Rightarrow \quad P^{00} = 0 \qquad P^{0i} = 0 \qquad P^{ij} = \delta_{ij} - h_{ij}$$

$$\Rightarrow \quad \Gamma^0_{00} = \Gamma^0_{0i} = 0 \qquad \Gamma^0_{ij} = \frac{1}{2}\,\partial_0\, h_{ij}$$

$$\nabla_0 u_0 = \nabla_0 u_i = 0 \qquad \nabla_i u_j = \frac{1}{2}\,\partial_0\, h_{ij} \qquad \nabla \cdot u = \frac{1}{2}\,\partial_0\, h_{ii} = 0$$

The fluid remains at rest but the covariant derivative of the velocity is non-zero

Assume that the only non-zero value of h_{ij} is h_{12}

$$\sigma^{12}(t) = \eta\,\partial_0 h_{12}(t) \quad \Rightarrow \quad \boxed{\sigma^{12}(\omega) = -i\eta\,\omega\, h_{12}(\omega)}$$

Linear response $\longrightarrow$ $\sigma^{12}(\omega) = G_R^{12,12}(\omega, \vec{k}=0)\, h_{12}(\omega)$

$\downarrow$

$$\eta = -\lim_{\omega\to 0}\left[\frac{1}{\omega}\,\mathrm{Im}G_R^{12,12}(\omega, \vec{k}=0)\right]$$

Kubo formula for the shear viscosity

Holographic calculation

Perturb the $(d+1)$-dimensional metric as:

$$ds^2 = g_{tt}\, dt^2 + g_{zz} dz^2 + g_{xx}\delta_{ij}\, dx^i dx^j$$

$\downarrow$

$$ds^2 = g_{tt}\, dt^2 + g_{zz} dz^2 + g_{xx}\big(\delta_{ij}\, dx^i dx^j + 2\phi\, dx^1 dx^2\big)$$

$g_{12} = g_{xx}\,\phi$ $\longrightarrow$ ${g^1}_2 = \phi$ $\longrightarrow$ ϕ is the source for T^{12} in the boundary

Action for ϕ

$$-\frac{1}{16\pi G_N}\int d^{d+1}x\, R \;\longrightarrow\; S_\phi = -\frac{1}{16\pi G_N}\int d^{d+1}x\,\sqrt{-g}\,\frac{1}{2}\,g^{MN}\,\partial_M\phi\,\partial_N\phi$$

ϕ behaves as a scalar field with $q = 16\pi\, G_N$ → $$\eta = \frac{1}{16\pi\, G_N} \frac{A_H}{V}$$

→ $$\frac{\eta}{s} = \frac{1}{4\pi}$$ $\left(\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\right.$ in ordinary units)

$k_B \rightarrow$ Boltzman constant

Numerically → $\frac{\eta}{s} = 0.07957$ → very small almost universal result at strong coupling

Finite coupling corrections → $$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{15\,\zeta(3)}{\lambda^{\frac{3}{2}}} + \cdots\right)$$ $\zeta(3) = 1.2020$

Perturbative result → $\eta/s \rightarrow \infty$ as $\lambda \rightarrow 0$

$$\frac{\eta}{s} = \frac{A}{\lambda^2 \log\left(\frac{B}{\sqrt{\lambda}}\right)}$$

$\mathcal{N} = 4$ SYM $\rightarrow$ $A = 6.174\,,\ B = 2.36$

QCD , $N_f = 0$ $\rightarrow$ $A = 34.8\,,\ B = 4.76$

Strongly coupled plasmas are almost perfect fluids

Kovtun, Son and Starinets (KSS) holographic bound $\rightarrow \eta/s \geq \frac{1}{4\pi}$

Not satisfied in higher derivative gravity

Example : Gauss-Bonnet gravity

$$S_{GB} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - 2\Lambda - \frac{3}{\Lambda} \lambda_{GB} \Big(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \Big) \right]$$

$$\downarrow$$

$$\boxed{\frac{\eta}{s} = \frac{1 - 4\lambda_{GB}}{4\pi}}$$

KSS bound violated if $\lambda_{GB} > 0$

Lowest η/s in Nature

- Quark-Gluon plasma at RHIC
- Ultracold atomic Fermi gases at very low T