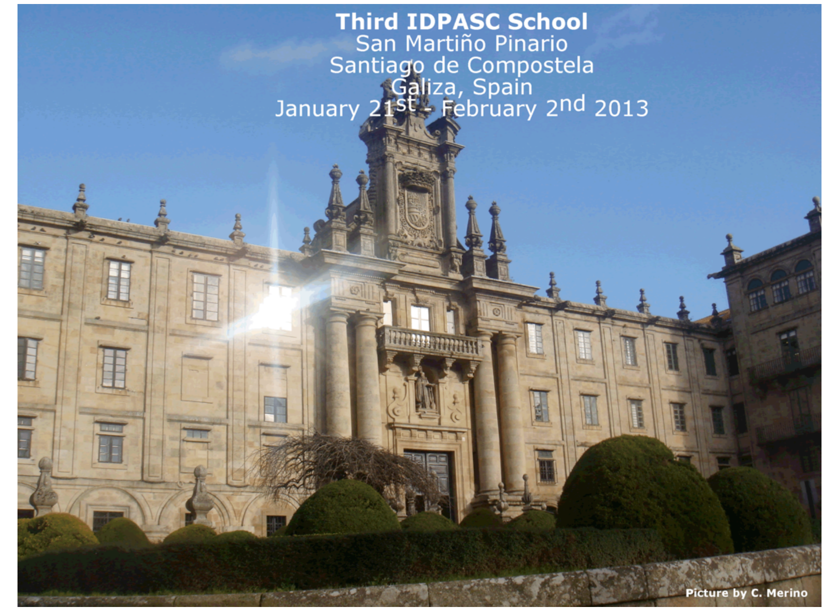

Introduction to AdS/CFT

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Non-perturbative structure of string theory

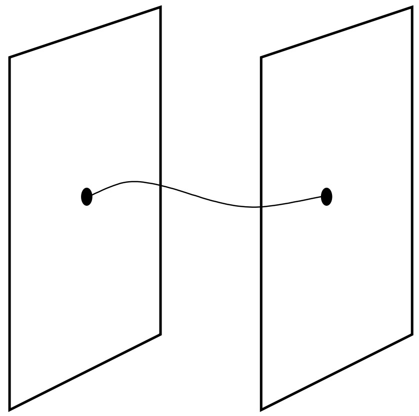
String theories have solitons $\longrightarrow$ extended objects

$\boxed{\text{Dp-branes}}$

Extended in $p+1$ directions (p spatial + time)

-Defined as hypersurfaces where strings end

-Obtained by quantizing the string with fixed ends along hyperplanes (Dirichlet boundary conditions). $\longrightarrow$



Charged under the antisymmetric tensor fields

$$A_{\mu_1\cdots\mu_{p+1}} \to \int_{\mathcal{M}_{p+1}} A_{\mu_1\cdots\mu_{p+1}}\, dx^{\mu_1}\cdots dx^{\mu_{p+1}}$$

$$\mathcal{M}_{p+1} \to \text{worldvolume of the Dp-brane}$$

The D-branes are dynamical objects

Action $\Rightarrow$ $$S_{Dp} = -T_{Dp} \int d^{p+1}x \, [\cdots]$$

$T_{Dp} \to$ tension of the Dp-brane $$T_{Dp} = \frac{1}{(2\pi)^p \, g_s \, l_s^{p+1}}$$

$$T_{Dp} \sim g_s^{-1} \to \text{non-perturbative objects}$$

Excitations of a D-brane

-Deformation of shape and rigid motion

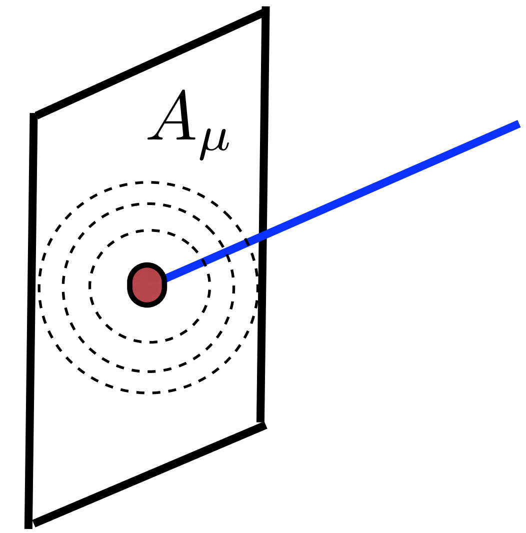
$$\text{Parametrized by } 9-p \text{ coordinates} \to \phi^i \;\; (i = 1, \cdots, 9-p)$$

$\Downarrow$

They are scalar fields on the worldvolume

-Internal excitations

The endpoint of the string is a charge that sources a gauge field on the worldvolume



Dirac-Born-Infeld action

$$S_{DBI} = -T_{Dp} \int d^{p+1}x \sqrt{-\det(g_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu})}$$

In flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + (2\pi l_s^2)^2 \, \partial_\mu \phi^i \, \partial_\nu \phi^i$$

Expanding in powers of $F_{\mu\nu}$ and ϕ

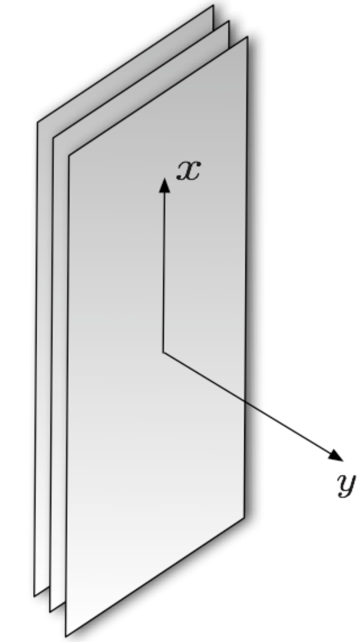
$$S_{DBI} = -\frac{1}{g_{YM}^2} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi^i \, \partial^\nu \phi^i + \cdots \right)$$

gauge+scalars+(fermions)

YM coupling

$$\boxed{g_{YM}^2 = 2(2\pi)^{p-2} \, l_s^{p-3} \, g_s}$$

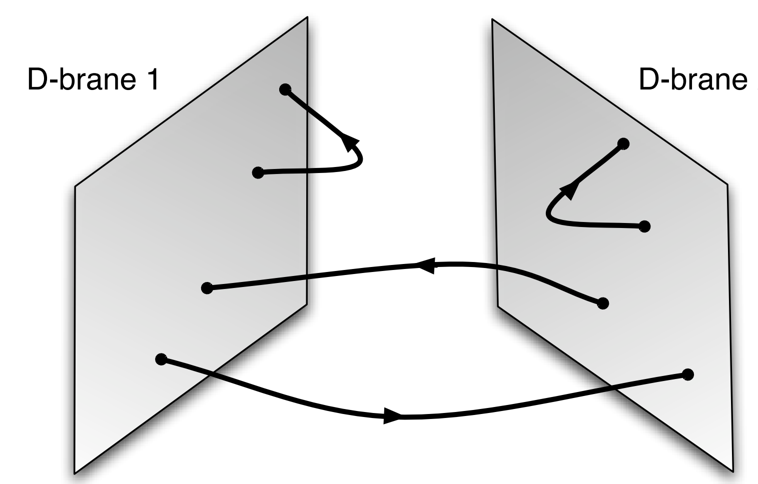
Multiple branes realize non-abelian gauge symmetry



N paralell coincident branes realize $U(N)$ YM

$A_\mu, \phi^i \rightarrow$ adjoints of $U(N)$

The non-abelian nature comes from strings stretched between different branes



The $U(1)$ can be decoupled

$$\boxed{\text{stack of } N \text{ Dp-branes} \Longrightarrow SU(N) \text{ gauge theory in } p+1 \text{ dimensions!}}$$

N D3-branes

-$3+1$ dimensional worldvolume
-$SU(N)$ gauge field
-$10-4=6$ scalars
-plus fermionic partners

$\rightarrow$ $\boxed{\mathcal{N}=4,\ SU(N)\ \text{SYM in}\ 4d}$

Exact CFT

$$\text{YM coupling} \rightarrow g_{YM}^2 = 4\pi g_s$$

The D-branes provide a completely new perspective on gauge theories

One can move the branes, put then in different spaces, .. $\rightarrow$ dualities, less SUSY, different field content &vacua,..

Geometric insight on gauge dynamics $\rightarrow$ brane engineering

String theory is a gravity theory $\longrightarrow$ any matter distorts the spacetime

The distortion is governed by the action

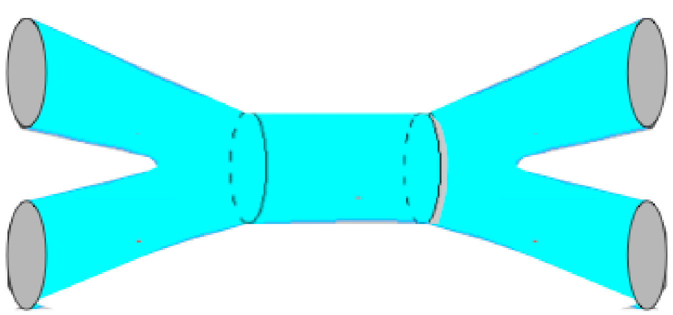
$$S = \frac{1}{16\pi G} \int d^{10}x \sqrt{-g}\, R + \cdots$$

10d Newton constant $\longrightarrow$ $\boxed{16\pi G = (2\pi)^7\, g_s^2\, l_s^8}$

-The dependence of G on l_s follows from dimensional analysis

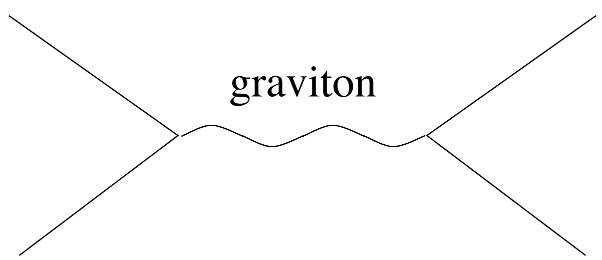
-The dependence on g_s $\rightarrow$ compare amplitudes in string theory and gravity

string theory



$\sim g_s^2$

gravity



$\sim G$

D-branes are solutions of Einstein equations

Linearized metric for a point-like object in D spacetime dimensions

$$ds^2 \approx -(1+2\varphi)\,dt^2 + \left(1 - \frac{2}{D-3}\varphi\right)\left(dx_1^2 + \cdots dx_{D-1}^2\right)$$

$$\varphi \sim \frac{GM}{r^{D-3}}$$ $\longrightarrow$ Newtonian potential

$$D-3 = d_T - 2$$

$$M \rightarrow \text{mass}$$

Linearized metric for an object extended along p spatial dimensions

$$ds^2 \approx (1+2\varphi)\left(-dt^2+dx_1^2+\cdots dx_p^2\right)+\left(1-\frac{2(p+1)}{D-p-3}\varphi\right)\left(dx_{p+1}^2+\cdots dx_{D-1}^2\right)$$

$$\varphi \sim \frac{GM}{r^{D-p-3}} \qquad d_T = D-1-p$$

Gravity solution for a stack of N D3-branes in 10d

$$ds^2 = H^{-\frac{1}{2}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_5^2\right)$$

$H(r) \rightarrow$ warp factor $\qquad H = 1 + \dfrac{L^4}{r^4}$

$dr^2 + r^2 d\Omega_5^2 \rightarrow$ flat metric of $\mathbb{R}^6$ $\qquad d\Omega_5^2 \rightarrow$ line element of $\mathbb{S}^5$

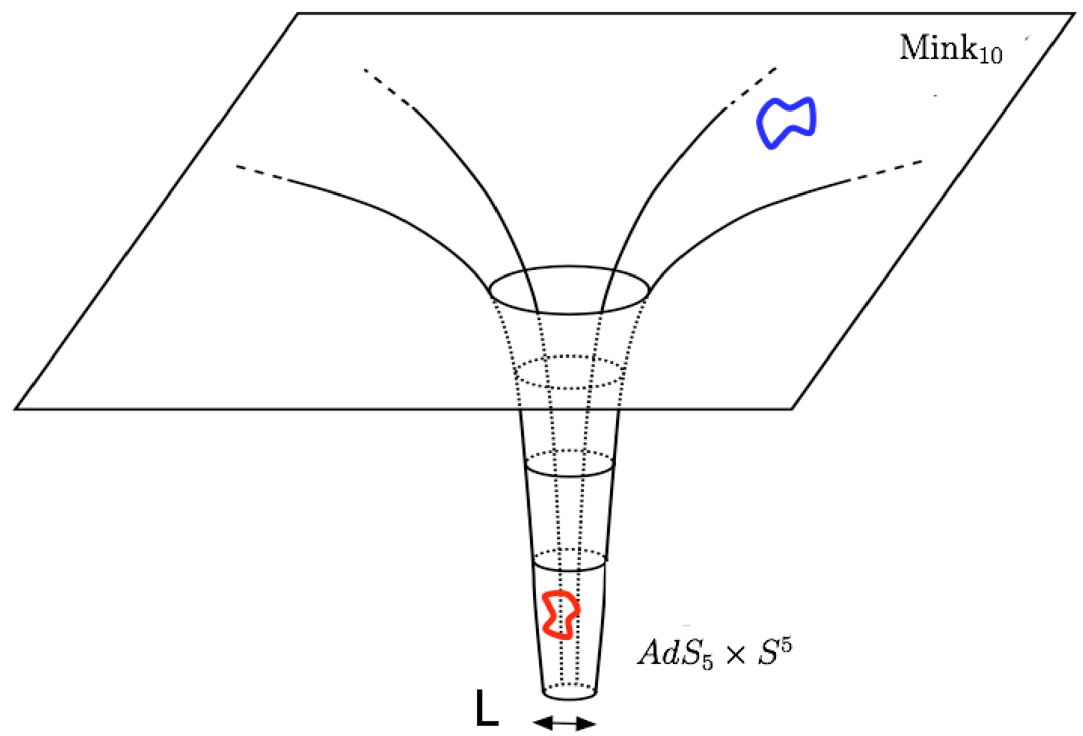
$$\boxed{L^4 = 4\pi g_s N l_s^4}$$

The value of L can be found at linearized level $\Longrightarrow$ $\varphi = -\dfrac{1}{4}\dfrac{L^4}{r^4}$

$G \sim g_s^2\, l_s^8$

$M \sim N\, T_{D3} \sim \dfrac{N}{g_s l_s^4}$ $\Longrightarrow$ $L^4 \sim GM$ $\Longrightarrow$ $GM \sim g_s^2\, l_s^8 \dfrac{N}{g_s l_s^4} \sim N g_s l_s^4$

D3-brane geometry $\rightarrow$ asymptotically 10d Minkowski with a infinite throat



near-horizon metric in the throat $\Rightarrow$ $H \approx \frac{L^4}{r^4}$ $\Rightarrow$ low-energy limit

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2}{r^2}\,dr^2 + L^2\,d\Omega_5^2$$

change variables as $\Rightarrow$ $r = \frac{L}{z}$

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2\right) + L^2\,d\Omega_5^2$$

near-horizon geometry $\rightarrow$ $AdS_5 \times \mathbb{S}^5$ with radius L

Maldacena conjecture

$$\boxed{\mathcal{N}=4\ SU(N)\ \text{SYM theory equivalent to string theory in } AdS_5\times S^5}$$

Relation of parameters

$$\left(\frac{L}{l_s}\right)^4 = N\, g^2_{YM}$$

't Hooft coupling $\longrightarrow$ $\boxed{\lambda = N\, g^2_{YM}}$ $\longrightarrow$ $\boxed{\frac{l_s^2}{L^2} = \frac{1}{\sqrt{\lambda}}}$

$$G = l_P^8 = \frac{\pi^4}{2}\, g^4_{YM}\, l_s^8 \longrightarrow \boxed{\left(\frac{l_p}{L}\right)^8 = \frac{\pi^4}{2N^2}}$$

The dual theory is classical gravity if:

$\longrightarrow$ $\frac{l_p}{L} << 1 \rightarrow$ no quantum gravity corrections

$\longrightarrow$ $\frac{l_s}{L} << 1 \rightarrow$ no stringy corrections

Equivalent to $N >> 1$ and $\lambda >> 1$ $\longrightarrow$ **planar strongly coupled SYM**

Checks of the correspondence $\Longrightarrow$ Symmetries on both sides

Conformal symmetry

- $\mathcal{N}=4$ SYM is a CFT with an exact vanishing β-function

- AdS_5 has the 4d conformal group $SO(2,4)$ as isometry group

dilatations $\Longrightarrow$ $(t,\vec{x}) \to \lambda(t,\vec{x}) \qquad z \to \lambda z$

Supersymmetry

- $\mathcal{N}=4$ SYM is maximally supersymmetric

32 fermionic supercharges $\to Q^A_\alpha,\, \bar{Q}^A_{\dot\alpha}, \qquad A = 1\cdots,4$

$Q^A \to$ rotated with $SU(4) \to R$-symmetry group $\Longrightarrow$ $Q^A_\alpha \to \mathbf{4} \quad \bar{Q}^A_{\dot\alpha} \to \bar{\mathbf{4}}$

It has six scalars $\phi_1,\cdots\phi_6 \to$ fundamental rep. $\mathbf{6}$ of $SO(6)\approx SU(4)$

- $AdS_5 \times \mathbb{S}^5$ is maximally supersymmetric

32 Killing spinors $\rightarrow$ supercharges of $\mathcal{N}=4$ SYM

rotational symmetry of $\mathbb{S}^5 \rightarrow SO(6) \rightarrow R$-symmetry of $\mathcal{N}=4$ SYM

directions along $\mathbb{S}^5 \rightarrow$ scalar fields on SYM

Relations of scales

$d \rightarrow$ proper distance on the bulk

$d_{YM} \rightarrow$ distance on the Minkowski coordinates

$$\boxed{d = \frac{L}{z} d_{YM}}$$

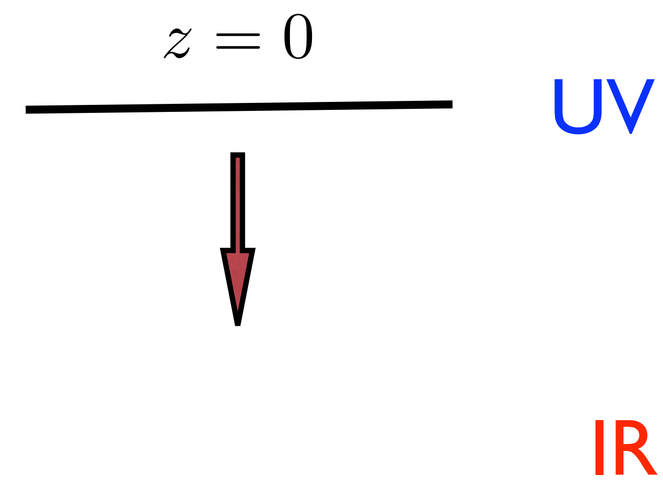
relation of energies $\Longrightarrow$ $\boxed{E = \frac{z}{L} E_{YM}}$

UV in field theory ($E_{YM} \rightarrow \infty$) $\rightarrow$ near-boundary region $z \rightarrow 0$

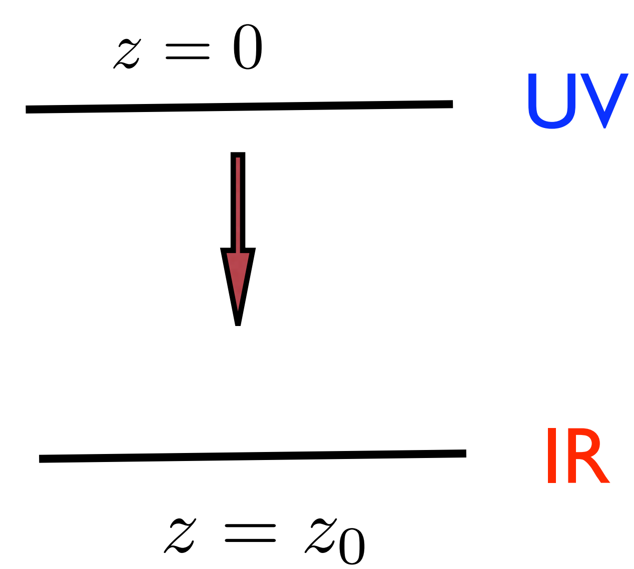
Field theory IR ($E_{YM} \rightarrow 0$) $\rightarrow$ near-horizon region $z \rightarrow \infty$

CFT $\rightarrow$ excitations at arbitrary low energies

geometry with bottomless throat



non-conformal theory $\rightarrow$ minimal scale $\rightarrow$ geometry ends smoothly at some z_0



- Confining theories with a mass gap m

$$z_0 \sim \frac{1}{m}$$

- Finite temperature theories with temperature T

$$z_0 \sim \frac{1}{T}$$

Scalar field in AdS_{d+1}

Euclidean metric $\Rightarrow$ $$ds^2 = \frac{L^2}{z^2}\left[dz^2 + \delta_{\mu\nu}\, dx^\mu\, dx^\nu\right]$$

action $\Rightarrow$ $$S = -\frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{MN}\,\partial_M\phi\,\partial_N\phi + m^2\,\phi^2\right]$$

eom $\Rightarrow$ $$\frac{1}{\sqrt{g}}\,\partial_M\left(\sqrt{g}\,g^{MN}\partial_N\,\phi\right) - m^2\,\phi = 0$$

$$\boxed{z^{d+1}\,\partial_z\left(z^{1-d}\,\partial_z\phi\right) + z^2\,\delta^{\mu\nu}\,\partial_\mu\partial_\nu\,\phi - m^2\,L^2\,\phi = 0}$$

Momentum space

$$\phi(z, x^\mu) = \int\frac{d^dk}{(2\pi)^d}\,e^{ik\cdot x}\,f_k(z)$$

$$\boxed{z^{d+1}\,\partial_z\left(z^{1-d}\,\partial_z f_k\right) - k^2\,z^2\,f_k - m^2\,L^2\,f_k = 0}$$

near the boundary $z = 0$ $\Longrightarrow$ $f_k \sim z^\beta$ $\Longrightarrow$ $\beta(\beta - d) - m^2 L^2 = 0$

Two solutions for β $\Longrightarrow$ $$\beta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2}$$

Then $\boxed{f_k(z) \approx A(k)\, z^{d-\Delta} + B(k)\, z^\Delta}$ $\boxed{\Delta = \frac{d}{2} + \nu}$ $\boxed{\nu = \sqrt{\frac{d^2}{4} + m^2 L^2}}$

In position space

$$\phi(z, x) \approx A(x)\, z^{d-\Delta} + B(x)\, z^\Delta$$

Δ is real if $\boxed{m^2 \geq -\left(\frac{d}{2L}\right)^2}$ Breitenlohner-Freedman (BF) bound

The term $z^{d-\Delta}$ dominantes as $z \to 0$

$$\phi(z=\epsilon, x) \approx \epsilon^{d-\Delta} A(x) \quad \Longrightarrow \quad \text{divergent if } m^2 > 0$$

Define the QFT source as $\boxed{\varphi(x) = \lim_{z\to 0} z^{\Delta-d} \phi(z,x)}$ $\Longrightarrow$ finite

Boundary action

$$S_{bdy} \sim \int d^d x \sqrt{\gamma_\epsilon}\, \phi(\epsilon, x)\, \mathcal{O}(\epsilon, x)$$

$$\gamma_\epsilon = \left(\frac{L}{\epsilon}\right)^{2d} \quad \Longrightarrow \quad S_{bdy} \sim L^d \int d^d x\, \varphi(x)\, \epsilon^{-\Delta}\, \mathcal{O}(\epsilon, x)$$

S_{bdy} finite if $\quad \mathcal{O}(\epsilon, x) = \epsilon^{\Delta}\, \mathcal{O}(x)$

$z = 0 \to z = \epsilon$ is a scale transformation in the QFT

$$\boxed{\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2 L^2}} \quad \Longrightarrow \quad \boxed{\Delta \to \text{mass scaling dimension of the dual operator } \mathcal{O}}$$

Inner product for solutions of the KG equation

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} dz\, d^d x \sqrt{-g}\, g^{tt} (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*) \qquad \Sigma_t \rightarrow \text{constant-}t \text{ slice}$$

- Subleading modes $B(x) z^{\Delta}$ $\Longrightarrow$ Normalizable

Elements of the Hilbert space of the boundary theory

- Leading modes $A(x) z^{d-\Delta}$ $\Longrightarrow$ Non-normalizable if $m^2 L^2 \geq -\frac{d^2}{4} + 1$

Sources of the boundary theory

Reduction on the $\mathbb{S}^5$

$\phi \longrightarrow$ field in $AdS_5 \times S^5$

$$\boxed{\phi(x,\Omega) = \sum_l \phi_l(x)\, Y_l(\Omega)}$$

$x \longrightarrow$ coordinates of AdS_5

$\Omega \longrightarrow$ coordinates of $\mathbb{S}^5$

$Y_l(\Omega) \longrightarrow$ spherical harmonics on $\mathbb{S}^5$

Reduced action

$$S = \frac{1}{16\pi G_5} \int d^5x \left[\mathcal{L}_{grav} + \mathcal{L}_{matter} \right] \quad \Longrightarrow \quad \mathcal{L}_{grav} = \sqrt{-g}\left[R + \frac{12}{L^2} \right]$$

Relation of Newton constants

$$\frac{1}{16\pi G} \int d^5x\, d^5\Omega \sqrt{-g_{10}}\, R_{10} \rightarrow \frac{L^5 \Omega_5}{16\pi G} \int d^5x \sqrt{-g_5}\, R_5 \quad \Longrightarrow$$

$$\boxed{G_5 = \frac{G}{L^5 \Omega_5} = \frac{G}{\pi^3 L^5} = \frac{\pi}{2N^2} L^3}$$

SYM central charge $\Longrightarrow$

$$\boxed{c_{SYM} = \frac{1}{4} \frac{L^3}{G_5} = \frac{N^2}{2\pi}}$$

$\phi \rightarrow$ massless scalar field in $AdS_5 \times S^5$

eom $\Rightarrow$ $\boxed{\nabla^2 \phi \; = \; 0}$ $\quad \nabla^2 = \nabla^2_{AdS_5} + \nabla^2_{\mathbb{S}^5}$

Eigenvalues of the laplacian on $\mathbb{S}^5$ $\Rightarrow$

$$\nabla^2_{\mathbb{S}^5} \, Y_l(\Omega) \; = \; -\frac{C_l^{(5)}}{L^2} \, Y_l(\Omega)$$

$$C_l^{(5)} = l(l+4) \qquad l = 0, 1, 2, \cdots$$

Eom of the reduced AdS_5 fields ϕ_l

$$\boxed{\nabla^2_{AdS_5} \, \phi_l = m_l^2 \, \phi_l} \qquad m_l^2 = \frac{l(l+4)}{L^2}$$

$\Rightarrow$ mass spectra after the reduction

scaling dimensions $\Rightarrow$ $\Delta = 2 + \sqrt{4 + (mL)^2}$ $\Rightarrow$ $\Delta = 2 + \sqrt{4 + l(l+4)}$

$$\boxed{\Delta = l + 4}$$

Field/operator dictionary

$\boxed{\text{massless case } l = 0}$ → $\Delta = 4$ ← dim[∂]=dim[A]=1

→ Scalar operator with dimension 4

→ Singlet under $SO(6)$

$\boxed{\mathcal{O} = \mathrm{Tr}\left[F_{\mu\nu} F^{\mu\nu}\right]}$ → glueball operator

$\boxed{\text{massive case } l \neq 0}$ → $\Delta = 4 + l$ ← dim[ϕ]=1

→ Transform as a symmetric tensor of $SO(6)$ with l indices

$\boxed{\mathcal{O}_{i_1,\cdots,i_l} = \mathrm{Tr}\left[\phi_{(i_1,\cdots,i_l)} F_{\mu\nu} F^{\mu\nu}\right]}$ → glueball dressed with scalars

$\phi_{(i_1,\cdots,i_l)} \rightarrow$ traceless symmetric product of l scalar fields

$\boxed{\text{The matching can be extended to all modes of 10d SUGRA on } AdS_5 \times S^5}$

Other fields

- $\boxed{\text{Antisymmetric fields with } p \text{ indices} \rightarrow A_{\mu_1 \cdots \mu_p}}$

$$\Delta \rightarrow \text{largest root of} \rightarrow \quad (\Delta - p)(\Delta + p - d) = m^2 L^2$$

$$\boxed{\Delta = \frac{d}{2} + \sqrt{\left(\frac{d-2p}{2}\right)^2 + m^2 L^2}}$$

- $\boxed{\text{spin } \tfrac{1}{2} \text{ field}}$

$$\boxed{\Delta = \frac{d}{2} + |m|}$$