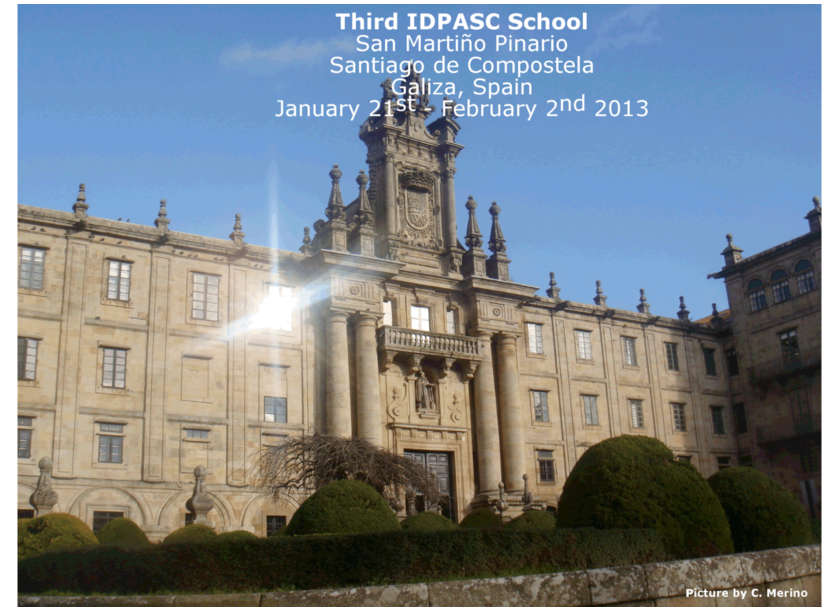

Introduction to AdS/CFT

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Lecture 1

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Main result: a duality relating QFT and gravity

Quantum Physics of strongly correlated many- body systems	⟷	Classical dynamics of black holes in one higher dimension
complicated		simple

CFT: conformal field theory

AdS: anti-de Sitter space

AdS/CFT is also known as:

- Holographic duality
- Gauge/gravity correspondence

In the spirit of condensed matter physics :

At strong coupling new weakly-coupled degrees of freedom emerge dynamically

New feature:

The emergent fields live in a space with one extra dimension $\longrightarrow$ holography

The extra dimension is related to the energy scale

The duality was obtained in the context of string theory

It has applications in:

- Strong coupling dynamics of gauge theories (QCD, integrability in QFT, electroweak symmetry breaking and LHC physics, string phenomenology,...)
- Condensed matter physics (holographic superconductors, quantum phase transitions, cold atoms, topological insulators,...)
- Black hole physics and quantum gravity
- Entanglement and quantum information theory
- Relativistic hydrodynamics

⋮

Here, we will concentrate on some concrete topics

Motivation from the renormalization group

Non-gravitational field theory for a lattice with lattice spacing a

$$H = \sum_{x,i} J_i(x)\, \mathcal{O}^i(x)$$

$x \to$ sites in the lattice

$i \to$ operators

$J_i(x) \to$ coupling constant (source) for the operator $\mathcal{O}^i(x)$

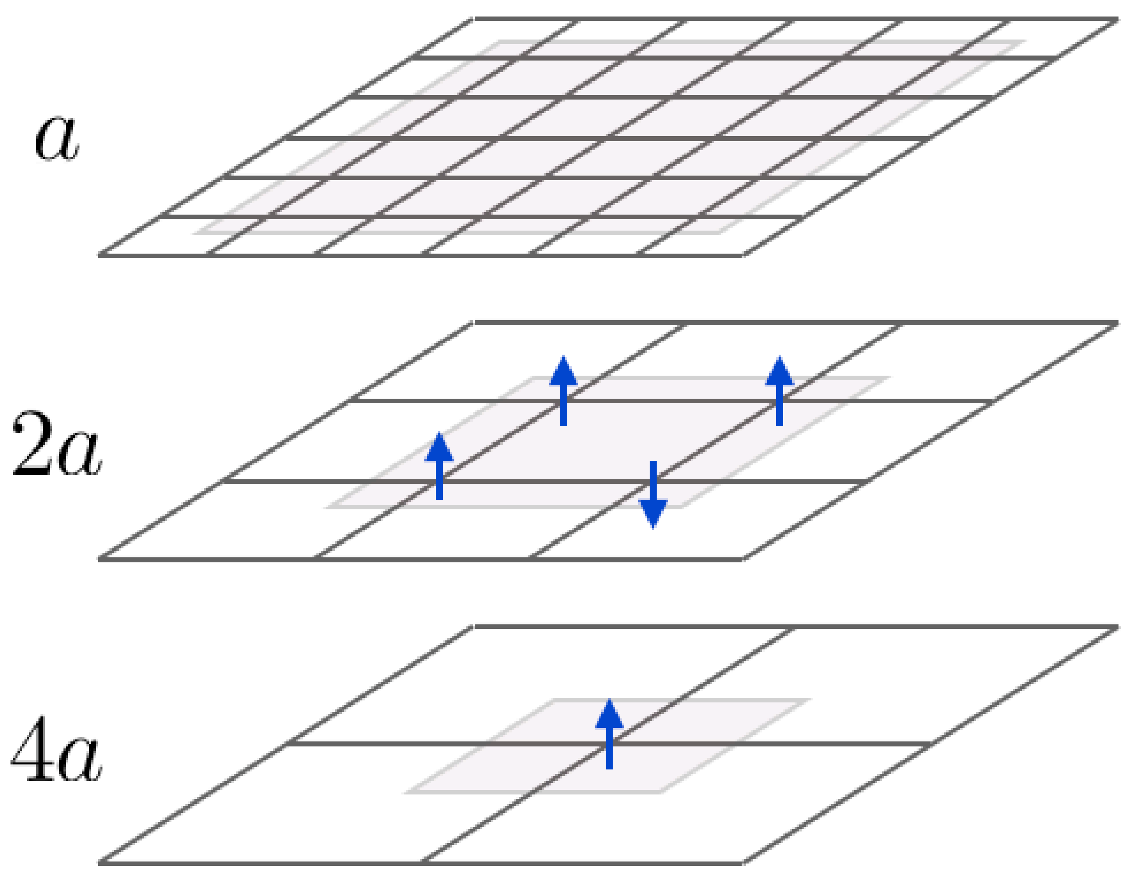
Kadanoff-Wilson Renormalization group

$\to$ Coarse grain the lattice by increasing the lattice spacing

$\to$ Replace multiple sites by a single site with the average value of the lattice variables

The couplings $J_i(x) \rightarrow$ change with the different steps:

$$J_i(x,a) \rightarrow J_i(x,2a) \rightarrow J_i(x,4a) \rightarrow \cdots$$



$$H = \sum_i J_i(x,a)\,\mathcal{O}^i(x)$$

$$H = \sum_i J_i(x,2a)\,\mathcal{O}^i(x)$$

$$H = \sum_i J_i(x,4a)\,\mathcal{O}^i(x)$$

The couplings are scale-dependent

$$J_i(x) \to J_i(x,u)$$

$u \to$ length scale at which we probe the system

$$u = (a, 2a, 4a, \cdots)$$

Coupling flow

$$\boxed{u\frac{\partial}{\partial u}\, J_i(x,u) \;=\; \beta_i\Big(J_j(x,u), u\Big)}$$

Weak coupling ⟶ β_i determined from perturbation theory

Strong coupling picture: ⟶ Think of u as an extra dimension

The multiple layers of lattices build up a new higher dimensional lattice
Regard the sources as fields in a space with one extra dimension

$$J_i(x,u) = \phi_i(x,r) \to \text{governed by some action}$$

AdS/CFT proposal:

$\boxed{\text{The dynamics of the sources } \phi_i(x,r) \text{ in the bulk is determined by gravity (i.e. some metric)}}$

It is a geometrization of the quantum dynamics encoded by the renormalization group

The sources must have the same tensor structure as the dual operators

$$\phi_i\, \mathcal{O}^i \text{ is a scalar}$$

-scalar field $\phi \to$ dual to a scalar operator $\mathcal{O}$

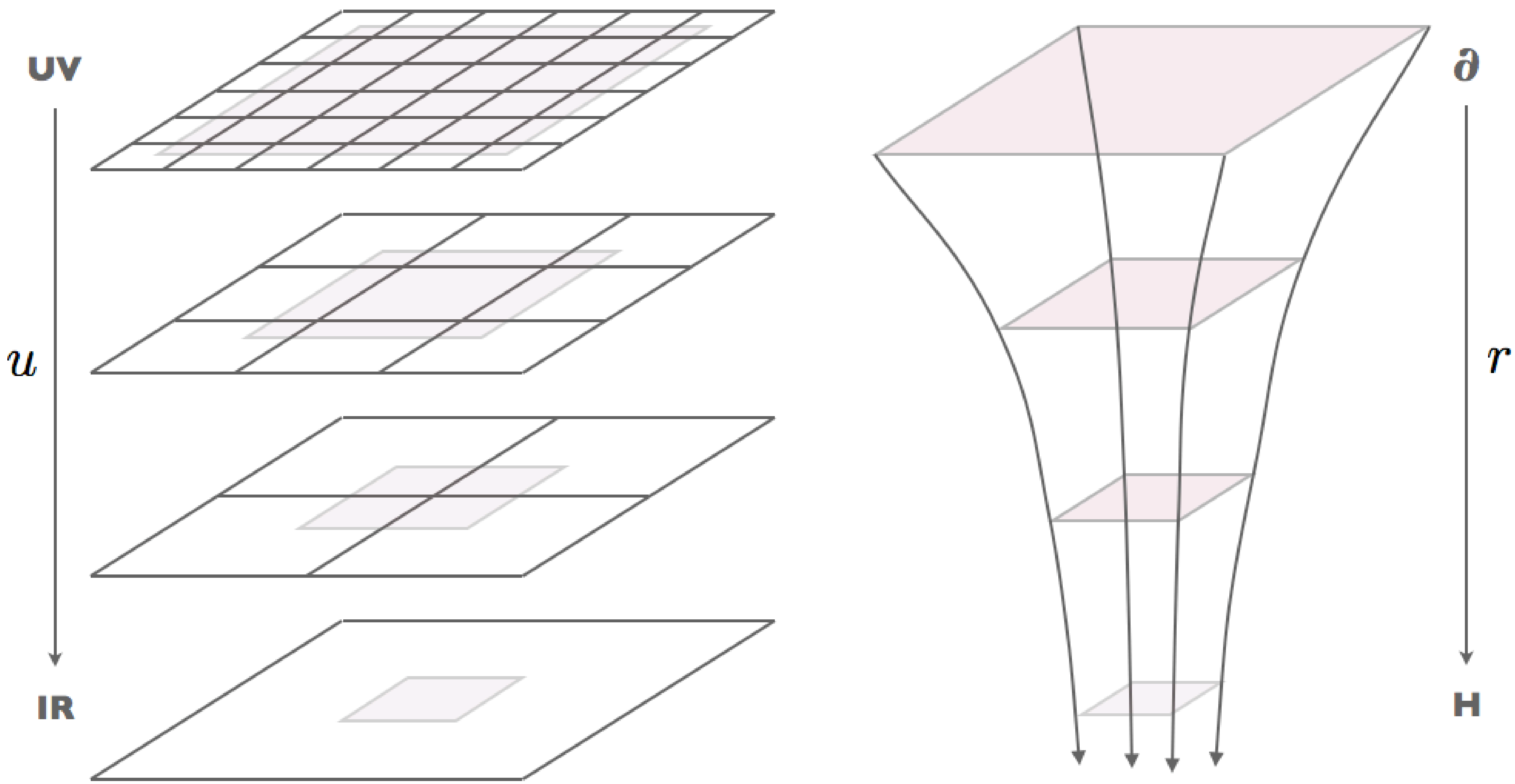
-vector field $A_\mu \to$ dual to a current J^μ

-spin-two field $g_{\mu\nu} \to$ dual to the energy-momentum tensor $T^{\mu\nu}$

Microscopic coupling in the UV $\Longrightarrow$ field of the gravity theory at the boundary

The field theory lives at the boundary of the higher dimensional space

$$J_i|_{UV} \quad = \quad \Phi_i|_\partial$$



Matching of the degrees of freedom

They are measured by the entropy

QFT side: The entropy is extensive

$R_d \to d$-dimensional region

$S_{QFT} \propto \mathrm{Vol}(R_d) \to$ proportional to the volume in d dimensions

Gravity side: The entropy in gravity is subextensive!

Bekenstein-Hawking formula:

$$\boxed{S_{BH} = \frac{A_H}{4G_N}}$$

$G_N \to$ Newton constant maximal entropy

Entropy in a volume $\leq$ entropy of a black hole inside the volume

$R_{d+1} \to$ region in $(d+1)$-dimensions $\qquad R_d = \partial R_{d+1}$

$$S_{GR}(R_{d+1}) \propto \mathrm{Area}(R_d) \propto \mathrm{Vol}(R_d)$$

Geometry at a fixed point $\Rightarrow$ vanishing β function $\Rightarrow$ CFT

Poincare invariant metric

$$ds^2 = \Omega^2(z)\left(-dt^2 + d\vec{x}^2 + dz^2\right)$$

$\vec{x} = (x^1, \cdots, x^{d-1})$

$z \longrightarrow$ extra dimension

Scale transformation

$$(t, \vec{x}) \longrightarrow \lambda(t, \vec{x}) \qquad z \longrightarrow \lambda z \qquad \lambda \longrightarrow \text{constant}$$

ds^2 invariant $\Rightarrow$ $\Omega(z) \to \lambda^{-1}\,\Omega(z)$ $\Rightarrow$ $\Omega(z) = \dfrac{L}{z}$

$$\boxed{ds^2 = \frac{L^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right)}$$

AdS boundary $\to z = 0$

UV of the QFT

Line element of AdS_{d+1}

Geometric interpretation of AdS

Consider

- Flat $(d+2)$-Minkowski spacetime with two times
- Coordinates $(y^{-1}, y^0, y^1, \cdots y^d)$

Metric $\Longrightarrow$ $$ds^2 = -(dy^{-1})^2 - (dy^0)^2 + \sum_{i=1}^{d} (dy^i)^2 \equiv \eta_{MN} dy^M dy^N$$

$$\boxed{AdS_{d+1} \Longrightarrow \text{Hyperboloid of radius } L \Longrightarrow \eta_{MN} y^M y^N = -L^2}$$

New coordinates

$$\boxed{\begin{array}{ll} u = y^{-1} + y^d & v = y^{-1} - y^d \\ x^\alpha = \dfrac{y^\alpha}{u} L = \dfrac{y^\alpha}{y^{-1} + y^d} L & \alpha = 0, \cdots, d-1 \end{array}} \Longrightarrow ds^2 = -du\, dv + \sum_{\alpha,\beta=0}^{d-1} \eta_{\alpha\beta}\, dy^\alpha\, dy^\beta$$

Hyperboloid eq. $\Longrightarrow$ $$-uv + \frac{u^2}{L^2} x^2 = -L^2$$

Induced metric on the hyperboloid $\Longrightarrow$ $z = \dfrac{L^2}{u}$ $\Longrightarrow$ $$ds^2 = \frac{L^2}{z^2} \left[dz^2 + \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta\right]$$

AdS_{d+1}

The AdS metric is a solution of the equation of motion of a gravity action of the type:

$$I = \frac{1}{16\pi G_N}\int d^{d+1}x\,\sqrt{-g}\,\Big[-2\Lambda + R + c_2 R^2 + c_3 R^3 + \cdots\Big]$$

$G_N \to$ Newton constant $\qquad \Lambda \to$ cosmological constant

$g = \det(g_{\mu\nu}) \qquad R = g^{\mu\nu}R_{\mu\nu} \to$ scalar curvature $\sim \partial\partial g$

$c_2 = c_3 = \cdots = 0 \to$ Einstein-Hilbert action of GR

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = -\Lambda g_{\mu\nu} \;\Longrightarrow\; R = g^{\mu\nu}R_{\mu\nu} = 2\,\frac{d+1}{d-1}\,\Lambda \;\Longrightarrow\; R_{\mu\nu} = \frac{2}{d-1}\,\Lambda\,g_{\mu\nu}$$

For AdS_{d+1} with radius L

$$\boxed{R_{\mu\nu} = -\frac{d}{L^2}\,g_{\mu\nu} \qquad \Lambda = -\frac{d(d-1)}{2L^2}}$$

Einstein space with negative cosmological constant

Counting the degrees of freedom (again)

$\boxed{\text{QFT side}}$

Put the system in a box of size R with lattice spacing ϵ

$$\boxed{N_{dof}^{QFT} = \left(\frac{R}{\epsilon}\right)^{d-1} c_{QFT}}$$

$c_{QFT} \to$ central charge $\Rightarrow$ # dof per cell

In $SU(N)$ YM $\to c_{SU(N)} \sim N^2$

$\boxed{\text{Gravity side}}$

Area of the AdS boundary

$$A_\partial = \int_{\mathbb{R}^{d-1},\, z=\epsilon} d^{d-1}x\,\sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{\mathbb{R}^{d-1}} d^{d-1}x \quad \Rightarrow \quad A_\partial = \left(\frac{RL}{\epsilon}\right)^{d-1}$$

$$\int_{\mathbb{R}^{d-1}} d^{d-1}x = R^{d-1}$$

Planck length and mass $\quad G_N = (l_P)^{d-1} = \dfrac{1}{(M_P)^{d-1}}$

From the entropy formula $\Rightarrow$ $\boxed{N_{dof}^{AdS} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_P}\right)^{d-1}}$

$$\boxed{\left(\frac{L}{l_P}\right)^{d-1} \sim c_{QFT}}$$

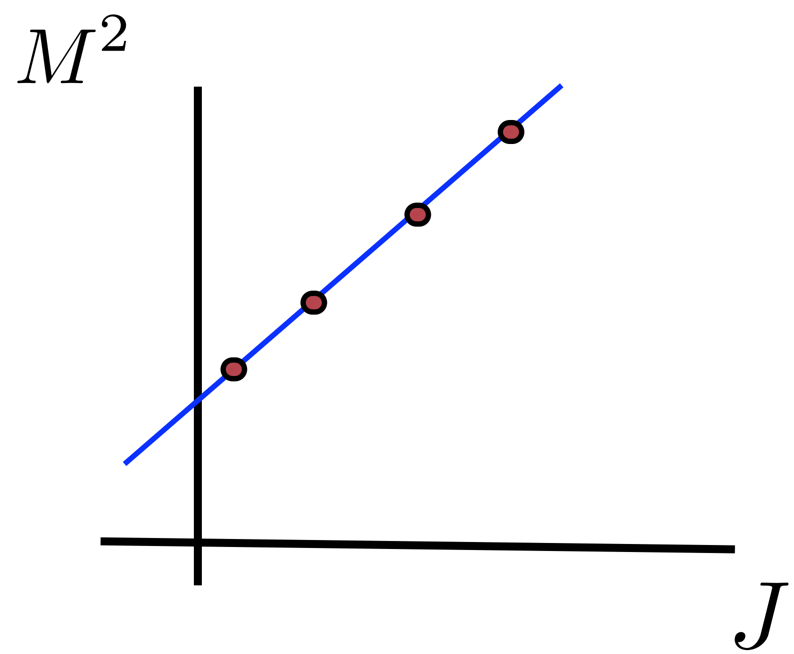
classical gravity in AdS $\sim \left(\frac{L}{l_P}\right)^{d-1} >> 1$

large # dof per unit volume

String theory basics

Historical origin → Description of hadronic resonances of high spin (60's)

Regge trajectories →



→ $\boxed{M^2 \sim J}$

Basic objects extended along some characteristic distance l_s

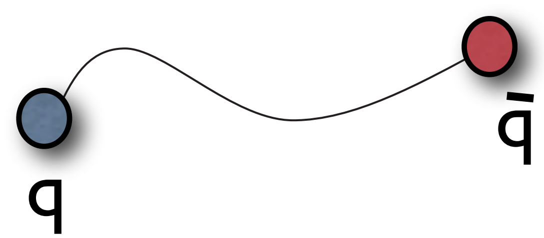
open

closed

→ non-local theory

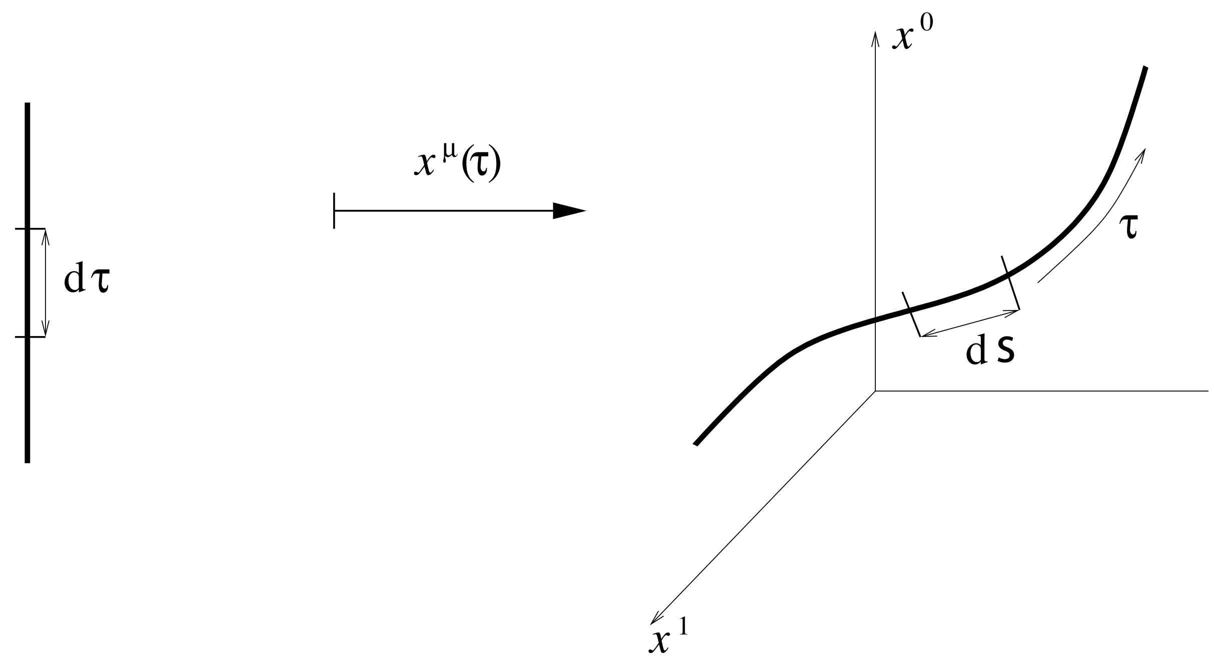
The rotational degree of freedom gives rise to high spins and Regge trajectories

In modern language a meson is a quark-antiquark pair joined by a string



$E \sim L$ → confinement!

Classical relativistic particle



$$x^\mu = x^\mu(\tau)$$

$x^\mu \longrightarrow$ parametrizes the space in which the point particle is moving

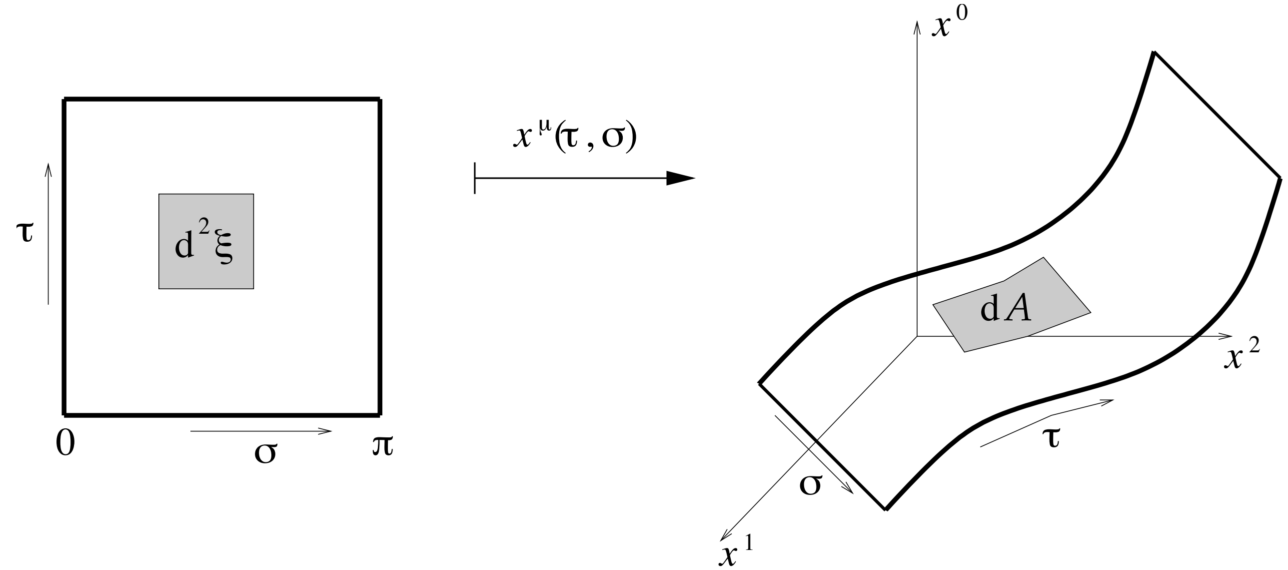
$\tau \longrightarrow$ world-line coordinate

Action →

$$\boxed{S = -m\int ds = -m\int_{\tau_0}^{\tau_1} d\tau\,\sqrt{-\eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu}}$$

Classical relativistic string

It describes a surface in spacetime ⟶ worldsheet



Nambu-Goto action

$$S_{NG} = -T \int dA \qquad T = \frac{1}{2\pi\, l_s^2} \qquad l_s \to \text{string length}$$

Worldsheet coordinates $\quad \xi^\alpha \quad \alpha = 0,1 \quad (\xi^0, \xi^1) = (\tau, \sigma)$

Embedding: $\Sigma \to \mathcal{M}$ with $\xi^a \to X^\mu(\xi^a)$

Induced metric ⟶ $\hat{G}_{\alpha\beta} \equiv G_{\mu\nu}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu$ ⟶

$$\boxed{S_{NG} = -T \int \sqrt{-\det \hat{G}_{\alpha\beta}}\; d^2\xi}$$

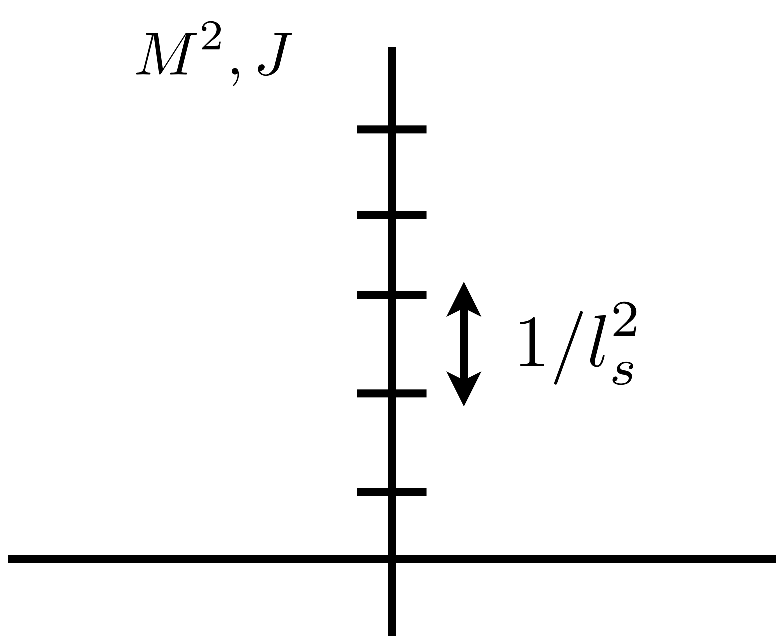
The classical eom's for the relativistic string can be solved in general for different boundary conditions (Neumann and Dirichlet)

String quantization

It opens Pandora's box!!

-Oscillation modes can be interpreted as particles

-Spectrum with infinite tower of particles with growing masses and spins



$\text{mass gap} \sim 1/l_s$

Consistency requires:

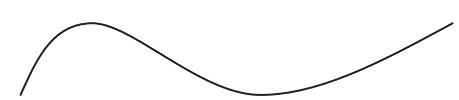
-Supersymmetry (symmetry between bosons and fermions)

If not there are tachyons in the spectrum (particles with $m^2 < 0$)

-The number of spacetime dimensions must be D=10

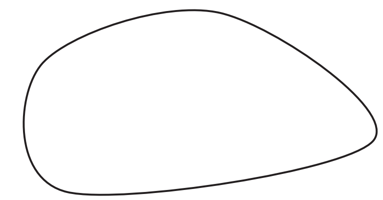
The extra dimensions should be regarded as defining a configuration space (as the phase space in classical mechanics)

Massless modes of the open string



Contains massless particles of spin one with the couplings needed to have gauge symmetry $\longrightarrow$ gauge bosons (fotons, gluons, ...)

Massless modes of the closed string



It contains a particle of spin 2 and zero mass which can be interpreted as the graviton (the quantum of gravity)

Moreover

Quantum consistency implies Einstein equations in 10d plus corrections:

$$R_{\mu\nu} + \cdots = 0$$

String theory is a theory of quantum gravity!!

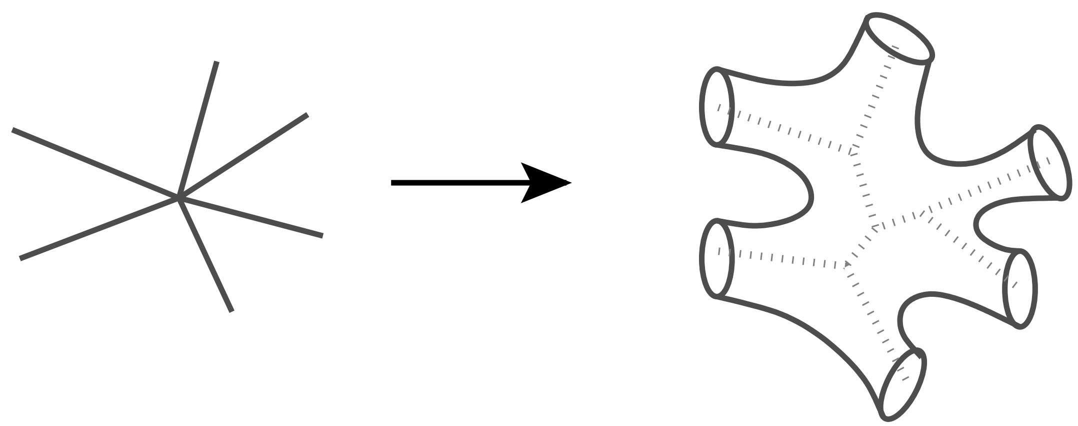
Thus $l_s \sim l_P$ (and not of the order of the hadronic scale ~ 1 fm)

Elementary strings with zero thickness were born for the wrong purpose

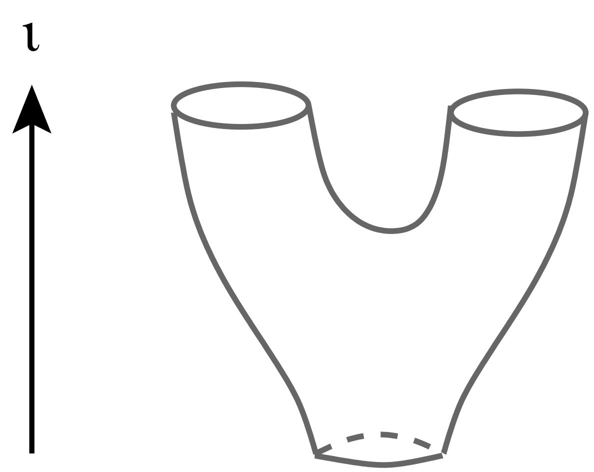
String interactions

Allow strings to join and split

It is like thickening a QFT vertex



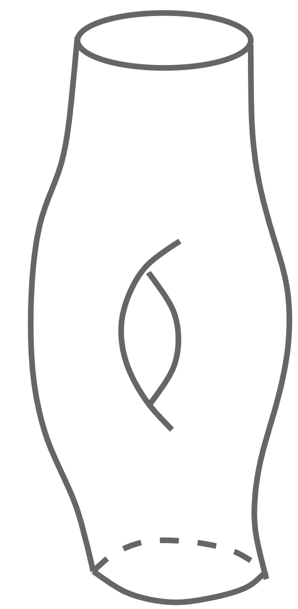
Triple vertex



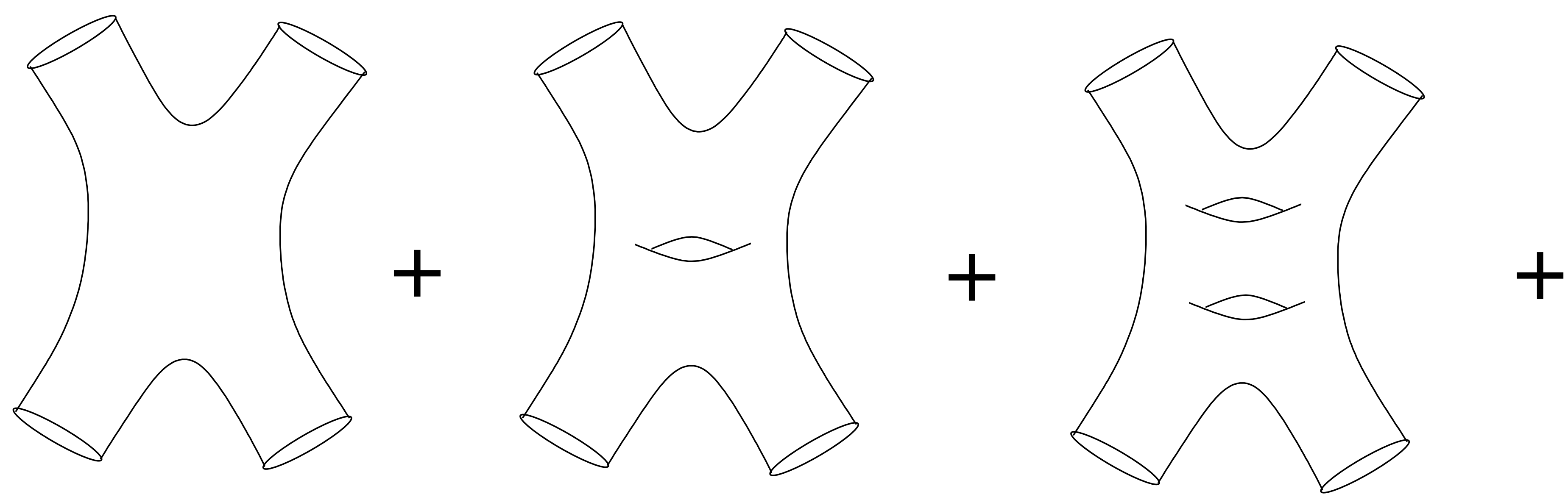
$\sim g_s$

$$\boxed{g_s \to \text{string coupling}}$$

Loop= hole in a Riemann surface



Perturbative series



A topological expansion!!

$$\boxed{\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2}\, F_h(\alpha')}$$

Topological expansion of gauge theories

$U(N)$ Yang-Mills theory $\Rightarrow$ $\mathcal{L} = -\frac{1}{g^2}\,\mathrm{Tr}\big[F_{\mu\nu}F^{\mu\nu}\big]$

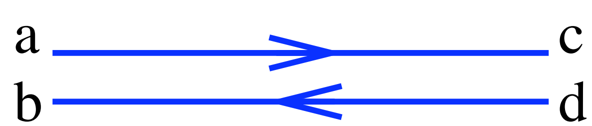
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ $\Rightarrow$ $A_\mu \to N \times N$ matrix $\Rightarrow$ $A^{a}_{\mu\,,\,b}$

Rewrite $\mathcal{L}$ $\Rightarrow$ $\boxed{\mathcal{L} = -\frac{N}{\lambda}\,\mathrm{Tr}\big[F_{\mu\nu}F^{\mu\nu}\big]}$ $\Rightarrow$ $\boxed{\lambda = g^2 N \to \text{'t Hooft coupling}}$

't Hooft topological expansion $\to$ in powers of N at fixed λ

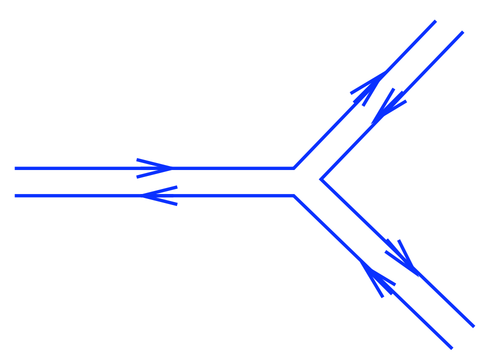
double line notation

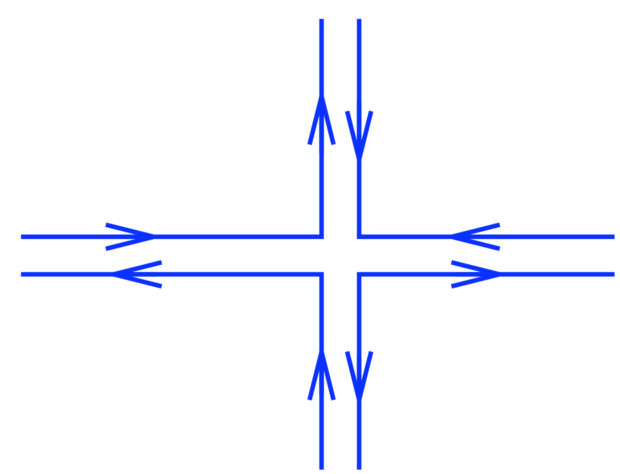
gauge propagator



$\sim \frac{\lambda}{N}$

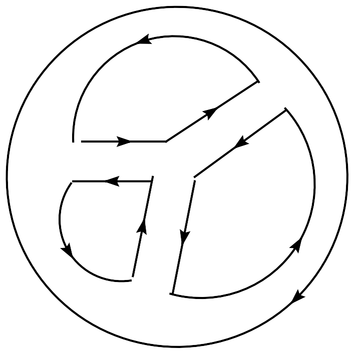
vertices





$\sim \frac{N}{\lambda}$

index loop $\to$ color sum $\to$ N



- The color lines form the perimeter of an oriented polygon (a face)
- Polygons join at a common edge

Every vacuum graph is associated to a triangulated 2d surface

Diagram D with

$E \rightarrow$ number of propagators (edges) connecting two vertices

$V \rightarrow$ number of vertices

$F \rightarrow$ number of index loops (faces) $\Longrightarrow$

$$\boxed{D \sim \left(\frac{\lambda}{N}\right)^E \left(\frac{N}{\lambda}\right)^V N^F = N^{F-E+V}\,\lambda^{E-V}}$$

Power of N $\Longrightarrow$ $F - E + V = \chi$ $\Longrightarrow$ Euler characteristic (topological invariant)

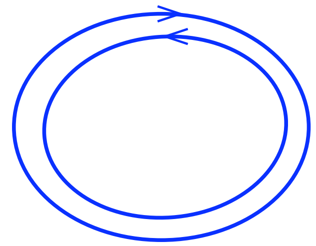
$\chi = 2 - 2h$ $\Longrightarrow$ $h \rightarrow$ genus $\rightarrow$ number of handles

Planar diagrams $\rightarrow h = 0 \rightarrow$ triangulate a sphere $\sim N^2$

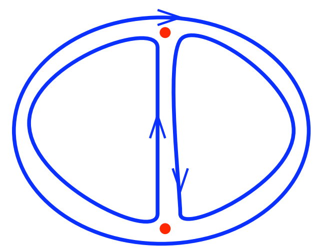
Non-planar diagrams $\rightarrow h \geq 1 \rightarrow h$-torus $\sim N^{2-2h}$

Examples

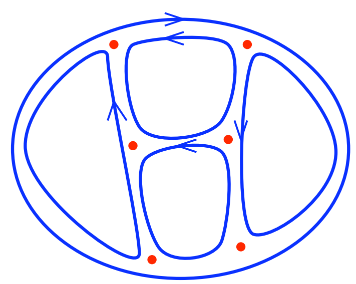
$\boxed{\chi = 2, \;\; h = 0}$ → planar diagrams



→ $E = 0, \;\; V = 0, \;\; F = 2$ → N^2

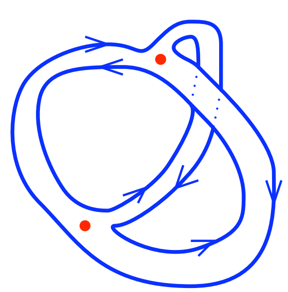


→ $E = 3, \;\; V = 2, \;\; F = 2$ → λN^2



→ $E = 9, \;\; V = 6, \;\; F = 5$ → $\lambda^3 N^2$

$\boxed{\chi = 0, \;\; h = 1}$ → non-planar diagrams



→ $E = 3, \;\; V = 2, \;\; F = 1$ → λN^0

Planar diagrams are dominant for large N and go like $N^2 \lambda^n$

Connected vacuum-to-vacuum amplitude

$$\boxed{\log Z = \sum_{h=0}^{\infty} N^{2-2h} \sum_{l=0}^{\infty} c_{l,h}\, \lambda^l = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)}$$

$f_h(\lambda) \rightarrow$ sum of diagrams that can be drawn in a surface of genus h

Similar to the perturbative expansion of string theory with

$$g_s \sim \frac{1}{N}$$

Heuristically: gauge theory diagrams triangulate the worldsheet of an effective string

The AdS/CFT correspondence is a concrete realization of this connection for $N, \lambda \rightarrow \infty$ (planar and strongly coupled)