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Compact objects in Einstein-Cartan theory: the effects of intrinsic spin in celestial bodies

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Introduction

The Einstein-Cartan theory differs from the theory of General Relativity by relaxing the imposition that the connection must be symmetric.

↳ The anti-symmetric part of the connection defines a tensor field usually dubbed torsion.

The extra structure in the manifold allows the formulation of local gauge theories. In particular it is possible to impose a local Poincaré gauge symmetry on the tangent space.

↳ This allow us to include the matter intrinsic spin in a geometric theory of gravity, relating it with the torsion tensor field.

Introduction

In the literature, the main line of study in the Einstein-Cartan theory and the inclusion of spin is how the latter may avoid the formation of singularities.

↳ So far there are no known suitable solutions for compact objects in the EC theory that can be smoothly matched to a vacuum exterior.

It was believed that in the cases of stars or even neutron stars, the corrections due to the spin-geometry coupling would be completely negligible.

↳ We will show that this is not the case. The presence of spin markedly changes the geometry of the space-time.

Setup

We propose to study static, spherically symmetric solutions for the Einstein-Cartan theory for a Weyssenhoff fluid source.

A Weyssenhoff fluid is a semi-classical model for a fluid composed of fermions.

In the spherically symmetric case, it is characterized by an energy-density, μ, pressure, p, and spin density, δ, with canonical energy-momentum and hypermomentum tensors:

$$\mathcal{T}_{\alpha\beta} = \mu\, u_\alpha u_\beta + p\left(g_{\alpha\beta} + u_\alpha u_\beta\right)$$

$$\Delta_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta} u^\gamma \delta$$

Setup

In the Einstein-Cartan theory, the space-time is characterized by the metric tensor, $g_{\alpha\beta}$, the torsion tensor

$$S_{\alpha\beta}{}^{\gamma} = C_{[\alpha\beta]}{}^{\gamma} \; ,$$

and a non-metricity tensor which we will impose to be null.

From the field equations, in the spherically symmetric case we find the following relation between the torsion tensor and spin density

$$S_{\alpha\beta}{}^{\gamma} = \varepsilon_{\alpha\beta} u^{\gamma} \delta$$

Setup

Finally, in a space-time with a general torsion the Weyl tensor can be written as

$$C_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}E^{\nu\mu} - 2u_\alpha E_{\beta[\gamma}\,u_{\,\delta]} + 2u_\beta E_{\alpha[\gamma}\,u_{\,\delta]} - \\ - 2\varepsilon_{\alpha\beta\mu}H_1^{\mu}{}_{[\gamma}\,u_{\,\delta]} - 2\varepsilon_{\mu\gamma\delta}H_2^{\mu}{}_{[\alpha}\,u_{\,\beta]}$$

where

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}\ u^\mu u^\nu$$

$$(H_1)_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\mu\nu\beta\delta}u^\delta$$

$$(H_2)_{\alpha\beta} = \frac{1}{2}\varepsilon_\alpha{}^{\mu\nu}C_{\beta\delta\mu\nu}u^\delta$$

Structure equations

Static, spherically symmetric space-times permeated by a Weyssenhoff fluid are completely characterized by the following set of variables

$$\{\mu, p, \delta, \phi, \mathcal{A}, \mathcal{E}, \mathrm{H}_1, \mathrm{H}_2, \mathcal{H}_1, \mathcal{H}_2\}$$

Verifying the evolution equations:

$$\hat{p} + \mathcal{A}\left(\mu + p\right) = -\frac{1}{4\pi}\delta\mathcal{H}_2\,,$$

$$\hat{\mathcal{A}} + \mathcal{A}\left(\mathcal{A} + \phi\right) + 2\delta^2 = 4\pi\left(\mu + 3p\right)\,,$$

$$\hat{\phi} + \frac{1}{2}\phi^2 + \mathcal{E} = -\frac{16\pi}{3}\mu\,,$$

$$\hat{\mathcal{E}} + \frac{3}{2}\mathcal{E}\phi + \delta\mathrm{H}_1 = \frac{8\pi}{3}\hat{\mu}\,,$$

$$2\hat{\delta} + \delta\phi = \mathrm{H}_1\,,$$

Structure equations

and constraint equations:

$$
\begin{aligned}
\mathcal{E} + \mathcal{A}\phi + 2\delta^2 &= \frac{8\pi}{3}(\mu + 3p) \;, \\
\mathcal{H}_1 &= \delta\phi \,, \\
\mathrm{H}_2 &= 2\delta\mathcal{A} - \delta\phi \,, \\
\mathrm{H}_1 + \mathrm{H}_2 + 2\mathcal{H}_2 &= 0 \,.
\end{aligned}
$$

Given an equation of state and a relation for the spin density, the previous set of equations completely describe the geometry of the space-time.

Junction conditions

We will be interested in studying solutions for compact objects, as such, in addition to the structure equations we need to determine the boundary conditions. This will be given by the junction conditions between an interior and exterior space-times.

Generalizing the Israel-Darmois formalism to space-times with torsion, we find, for the considered setup

$$[\phi]_{\pm} = 0\,, \qquad [\mathcal{A}]_{\pm} = 0\,,$$
$$[p]_{\pm} = 0\,, \qquad [\delta]_{\pm} = 0\,.$$

These conditions impose heavy constraints on the allowed interior solutions if the outside space-time is vacuum.

Maximum compactness of a star

Consider an interior static, spherically symmetric space-time permeated by a Weyssenhoff fluid, smoothly matched to an exterior Schwarzschild space-time. Assuming the quantity

$$\bar{\mu} = 8\pi\,\mu - \delta^2\,,$$

is non-negative and is a monotonically decreasing function in the radial coordinate r.

Defining r_0 the value of the circumferential radius of the matching surface, then

$$\frac{M\left(r_0\right)}{r_0} \leq \frac{4}{9}$$

where

$$M\left(r_0\right) = \frac{1}{2}\int_0^{r_0} \bar{\mu}\, r^2 dr$$

Exact solutions with vacuum exterior

Given an equation of state and a relation for the spin density we can solve the previous structure equations for an interior space-time.

Imposing $\delta^2 = \gamma\mu$ a possible interior solution is given by

$$ds^2 = -A\,(w)\,dt^2 + B\,(w)\,dw^2 + C\,(w)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

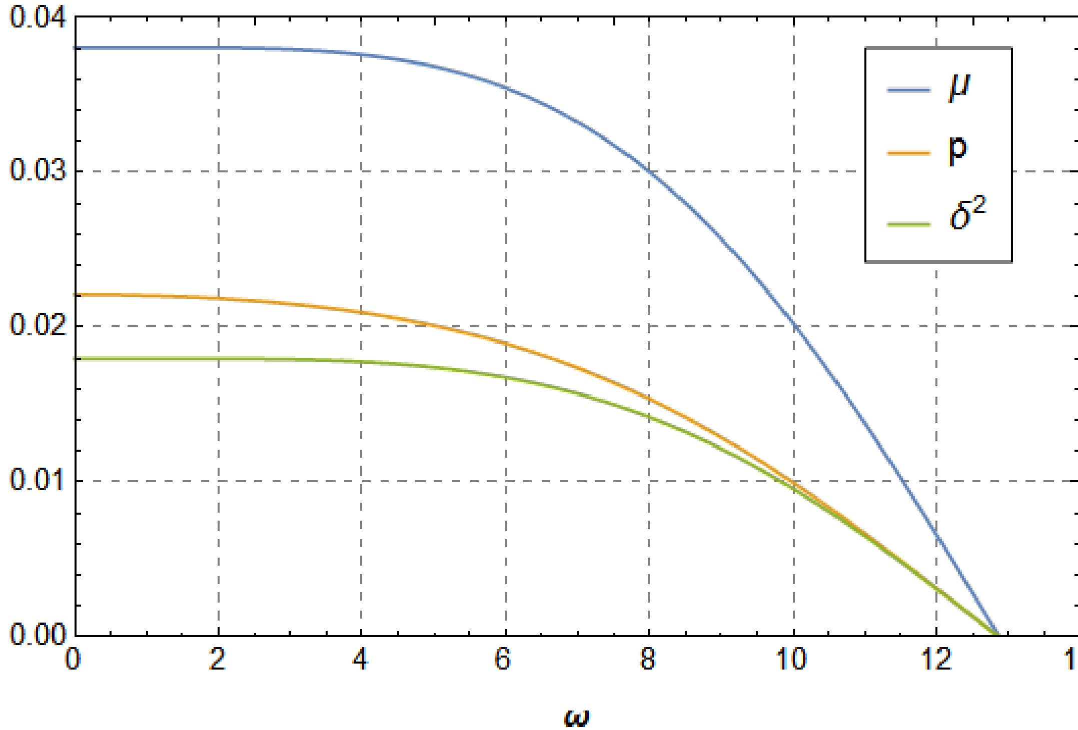
$$\eta = \frac{(a-1)\sin\,(R\,w)}{R\,w}\,, \qquad A = \frac{a\,(1+a+\eta)}{1+a+\eta}\,,$$

$$B = \frac{1+a+\eta}{a\,(1+a-\eta)}\,, \qquad C = \frac{w^2\,(1+a+\eta)^2}{4a^2}\,.$$

Exact solutions with vacuum exterior

with mass-energy density and pressure given by

$$\mu = \frac{aR^2\eta\left(3\eta - 2a - 2\right)}{\left(\gamma - 1\right)\left(1 + a + \eta\right)^2} \qquad p = \frac{aR^2\eta\left[2\gamma\left(2\eta - a - 1\right) - \eta\right]}{\left(\gamma - 1\right)\left(1 + a + \eta\right)^2}$$



No go result for spin held stars

As a final result, we considered a system where the interior solution would be supported by spin, that is, the case when $p = 0$.

Such system would be expected to model cold neutron stars, where the thermodynamical pressure tends to zero.

Theorem: *There are no static, spherically symmetric solutions of the Einstein-Cartan theory, sourced by a Weyssenhoff fluid such that:*

- *there are no horizons nor curvature singularities;*
- *The radial component of the acceleration is smooth for all* $r \in [0, r_0]$*;*
- *the spin and energy density functions:* δ *and* μ*, are at least of class* C^1*;*
- $\delta(r)$ *is non-null for all* $r \in [0, r_0[$ *and goes to zero for* $r = r_0$*;*
- $\delta^2(r)$ *is a monotonically decreasing function for all* $r \in [0, r_0]$*;*
- *the fluid's pressure,* p*, is null for all* $r \in [0, r_0]$*.*