Hands in cosmology

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Distances in the Universe

Open this tool:
`http://www.astro.ucla.edu/~wright/ACC.html`

Consider flat universe, $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_m = 0.31$, $T_0 = 2.725$ K, massless neutrinos

1. How old is an object which was created at $z = 1$?
2. How old was the universe when the oldest known galaxy ("GN-z11", $z = 11.1$) emitted the light we observe now?
3. What's the angular size distance D_A at photon decoupling?
4. What's the angle subtended by an object of 5 kpc if it is located at $z = 0.1$? (remember the definition of D_A)

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Consider flat universe, $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_m = 0.31$, $T_0 = 2.725$ K, massless neutrinos

1. How old is an object which was created at $z = 1$? — 7.635 Gyr
2. How old was the universe when the oldest known galaxy ("GN-z11", $z = 11.1$) emitted the light we observe now? — 389 Myr
3. What's the angular size distance D_A at photon decoupling? — 12.051 Mpc
4. What's the angle subtended by an object of 5 kpc if it is located at $z = 0.1$? (remember the definition of D_A) — $\theta = d/D_A$, 1.3×10^{-5} rad

Probability

[Trotta, arxiv:0803.4089]

What is probability?

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What is probability?

a frequency

"the number of times
the event occurs over
the total number of trials, in
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"randomness" of the trial series

what is really "random"?

do we properly know the initial
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Advantages:

- recovers frequentist on the long run;
- can be applied when frequentist cannot;
- no need to assume a distribution of possible data;
- deals effortlessly with nuisance parameters (*marginalization*);
- relies on *prior information*.

Bayes' theorem

how to deal with Bayesian probability?

given hypothesis H, data d, some information I (true):

Bayes theorem:

$$p(H|d,I) = \frac{p(d|H,I)\,p(H|I)}{p(d|I)}$$

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$\pi(\theta)$

Prior probability: what we knew before

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Posterior probability: what we know after

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sampling distribution of data, given that H is true

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or "Bayesian evidence",

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parameter inference

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$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

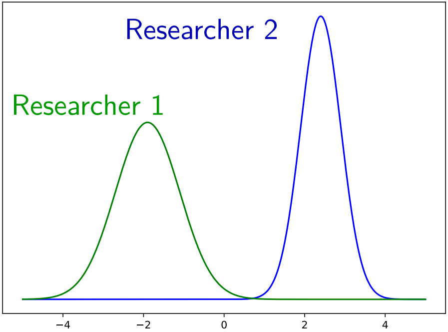
posterior → parameter inference

evidence → model comparison

Bayes theorem in action

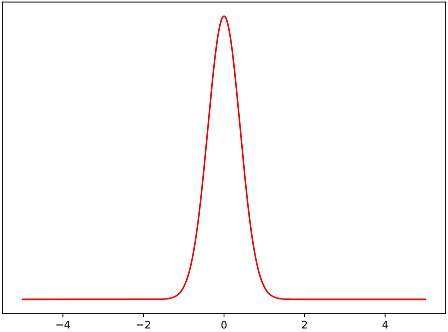
$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

Prior



What each researcher knew before the experiment

Likelihood

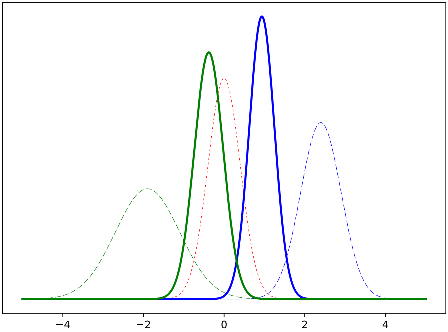


The result of the experiment

Bayes theorem in action

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

Posterior



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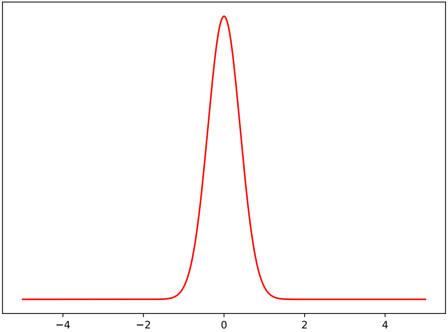

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Posterior depends on prior!

Bayes theorem in action

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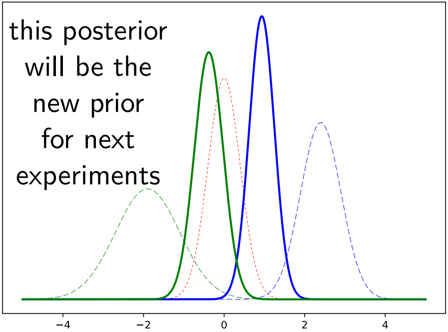

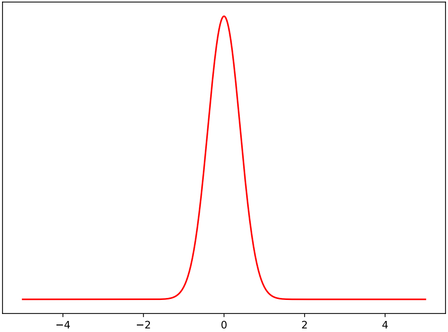

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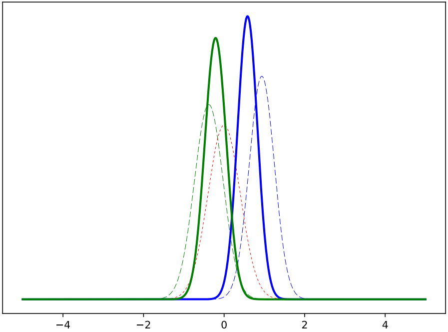
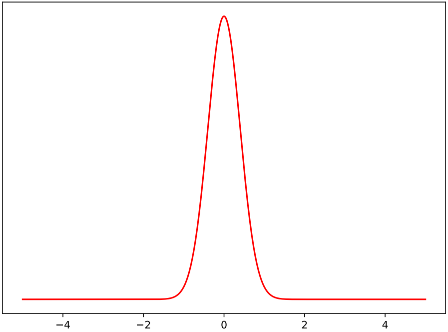
Posterior



What each researcher knows after the second experiment

Likelihood



The result of the experiment

Remember:
$\sigma_N^2 = \sigma^2/N$

Posterior depends on prior!

Bayes theorem in action

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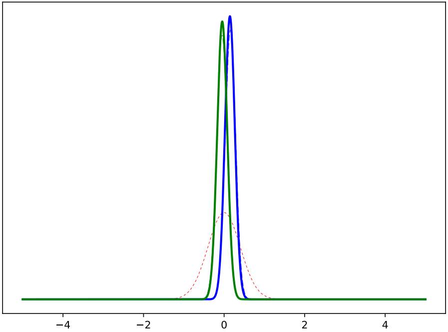
Posterior

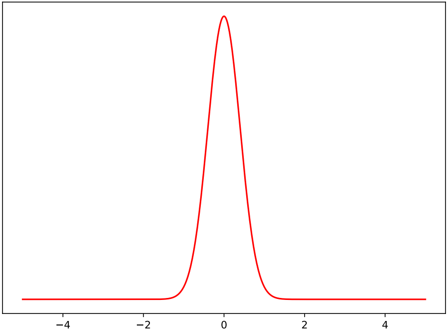


What each researcher knows after 10 experiments

Remember:

$\sigma_N^2 = \sigma^2/N$

Likelihood



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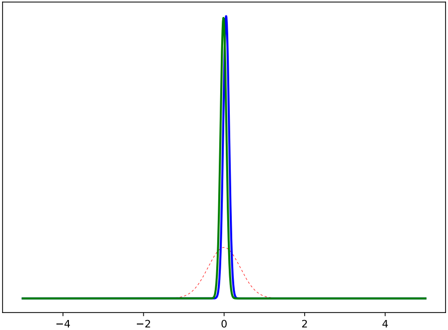
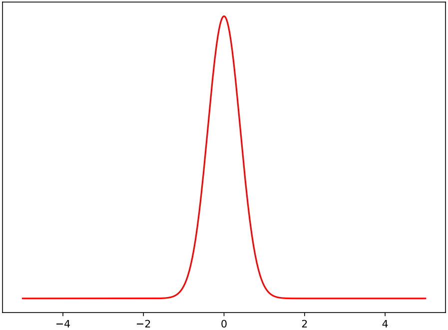


What each researcher knows after 30 experiments

Remember:

$\sigma_N^2 = \sigma^2/N$

Likelihood



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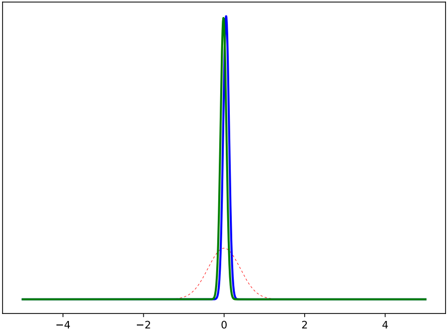

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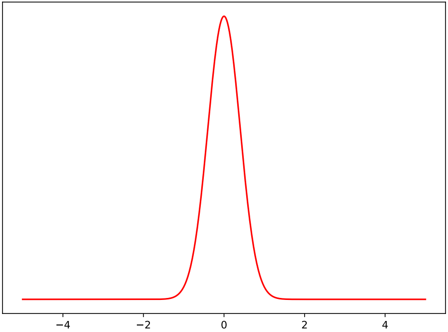

Knowledge converges using information from experiments

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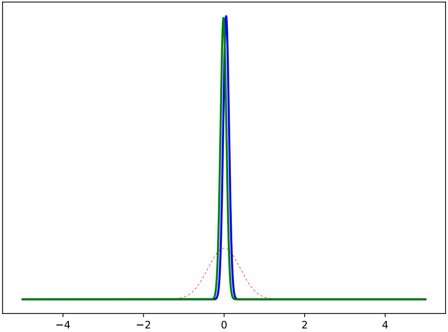

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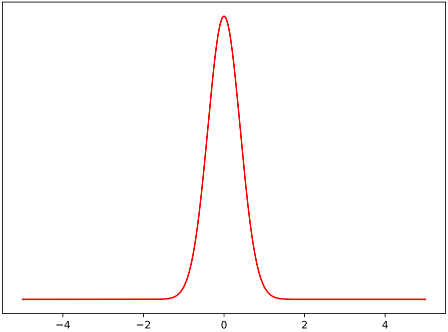

Knowledge converges using information from experiments

Prior dependence (subjectivity) only if not enough information in data!

(Bayesian) Parameter inference

Parameter inference = what we learn on the parameters, given:

data d

model $\mathcal{M}_0$

build $\mathcal{L}(\theta)$ from d & $\mathcal{M}_0$

parameters θ (& priors)

(physical ϕ + nuisance ψ)

Full posterior:

$$p(\theta|d,\mathcal{M}_0) \propto \mathcal{L}(\theta) \times p(\theta|\mathcal{M}_0)$$

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Marginalize over nuisance to obtain posterior for physical:

$$p(\phi|d,\mathcal{M}_0) \propto \int \mathcal{L}(\phi,\psi)p(\phi,\psi|\mathcal{M}_0)d\psi$$

marginalize over all the parameters except one (two)

→ 1D (2D) posterior

Credible intervals from the posterior

Credible interval α?

range of values within which an unobserved parameter value falls with a particular subjective probability α

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Analogous to frequentist confidence intervals α

Bayesian credible interval:

- bounds as fixed;
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Frequentist confidence interval:

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- equal-tailed interval: same probability of being below or above the interval
- interval for which the mean is the central point

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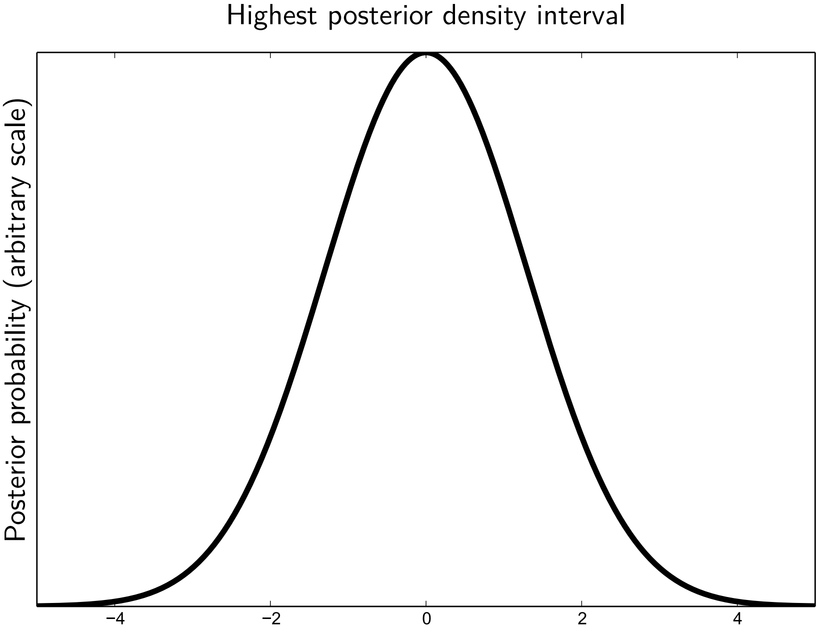

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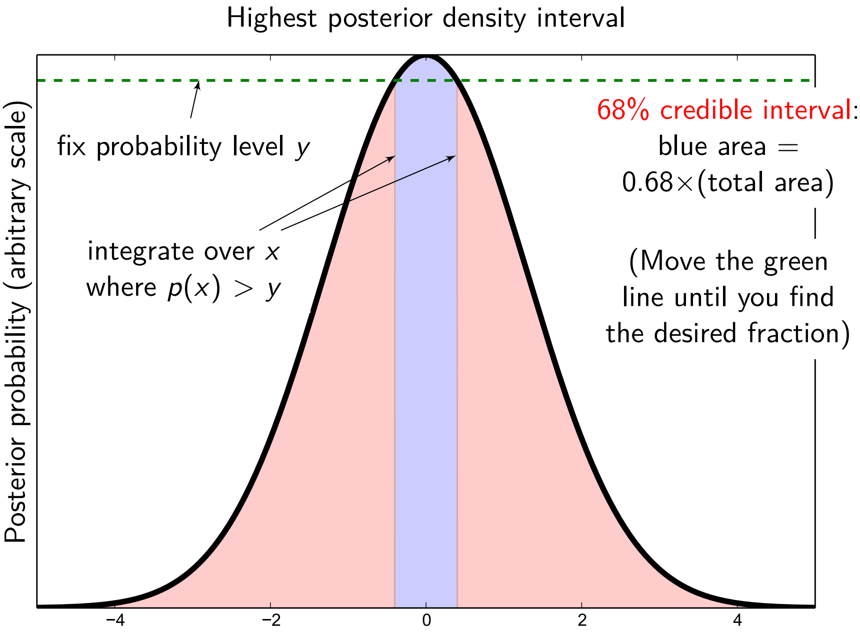

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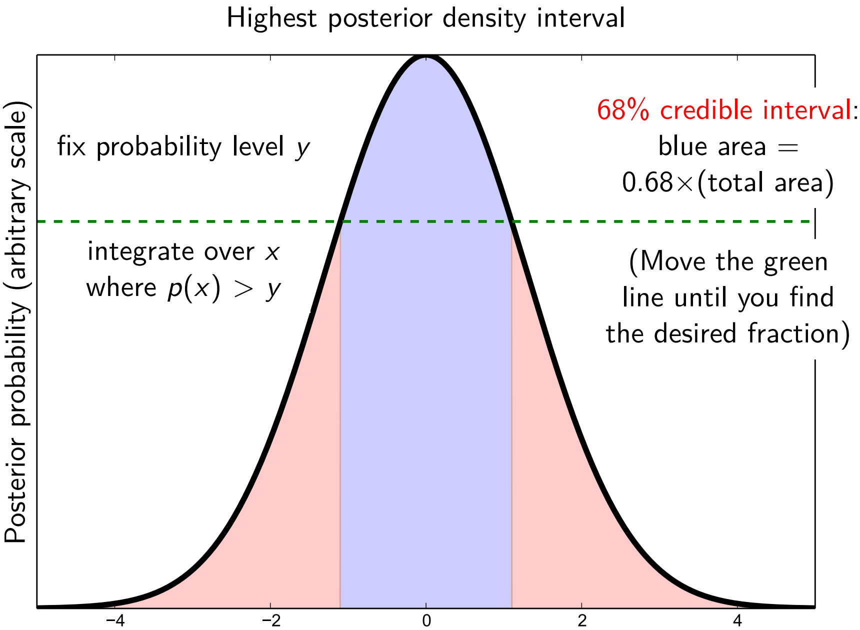

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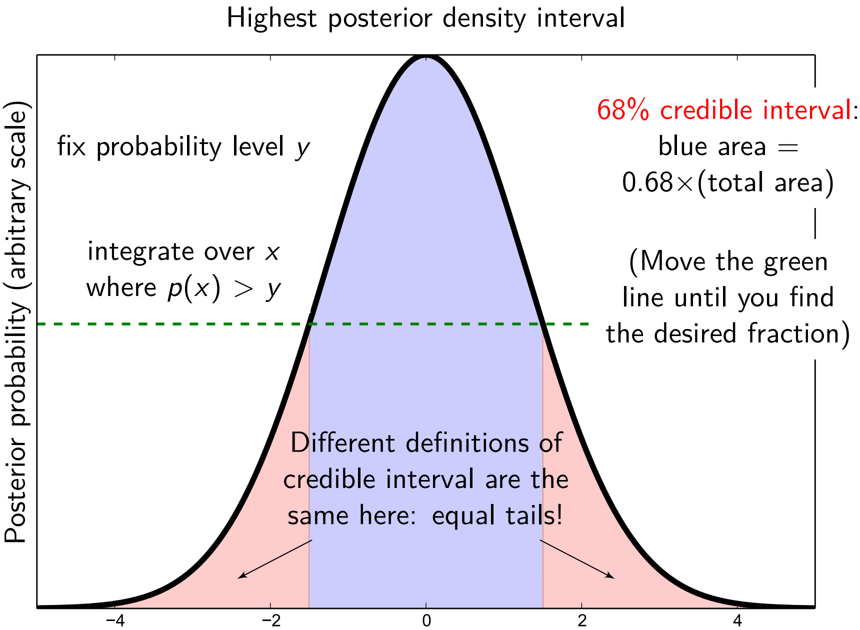

A simplified application to determine H_0

Idea from [Nature 551 (2017) 85-88 [arxiv:1710.05835]]

remember: $H_0 = v_H/d$

Data:

$d = 43.8^{+2.9}_{-6.9}$ Mpc,
approximate to
$d = 41.8 \pm 4.9$ Mpc

(obtained from GW signal)

$v_H = 3017 \pm 166$ km/s

(from GW signal+EM counterpart)

Questions:

1. best estimate for H_0?
2. posterior for H_0?
3. $1/2/3\sigma$ intervals on H_0?

$1\sigma = 68.3\%$ CL, $2\sigma = 95.5\%$ CL, $3\sigma = 99.7\%$ CL

A simplified application to determine H_0 - solutions

1. $H_0 \simeq 68.9$ km/s/Mpc from central values

2. use priors:

   $$\pi(d) \propto \exp\left[-\frac{1}{2}\left(\frac{(d-41.8)}{4.9}\right)^2\right]$$

   $$\pi(v_H) \propto \exp\left[-\frac{1}{2}\left(\frac{(v_H-3017)}{166}\right)^2\right]$$

   $$\pi(H_0) \propto 1$$

   use likelihood:
   $\mathcal{L}(v, d, H_0) \propto \delta(H_0, v_H/d)$

   marginalized posterior on H_0:

   $$p(H_0) = \int \mathcal{L}(v, d, H_0)\, p(d)\, p(v_H)\, p(H_0)\, \mathrm{d}d\, \mathrm{d}v_H$$

3. find the various levels, e.g.:
   68.27%: cut at $p = 3.069$, obtain
   $H_0 \in [62.7, 81.8] \rightarrow 71.2^{+10.6}_{-8.5}$ km/s/Mpc

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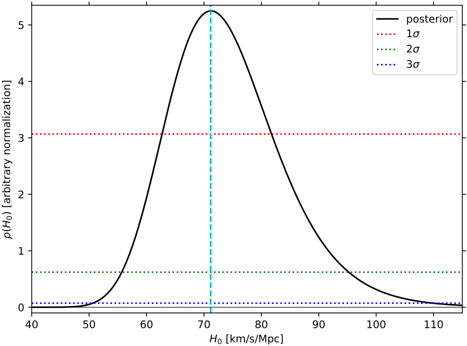

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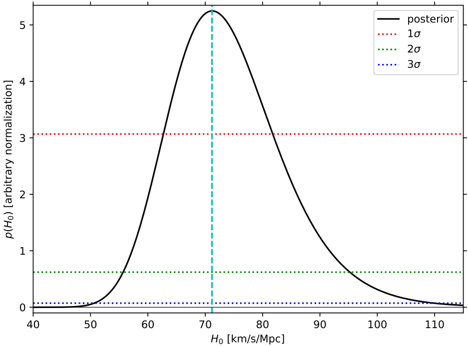

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1 *Distances*

2 *Bayesian statistics and GW*

- Some introduction
- The physics case

3 ***Thermodynamics in the early universe***

Number densities

1

Consider a particle with mass m, degeneracy $g = 2$ and zero chemical potential.

What is its number density in the limits of large and small m/T?

Assume the three cases:

- Maxwell-Boltzmann, $f(p) \propto e^{-E/T}$
- Fermi-Dirac, $f(p) = \left(e^{E/T} + 1\right)^{-1}$
- Bose-Einstein, $f(p) = \left(e^{E/T} - 1\right)^{-1}$

Remember: $$n = g \int \frac{d^3p}{(2\pi)^3} f(p)$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x - 1} = \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x + 1}$$

$$\zeta(2) = \pi^2/6\,, \qquad \zeta(3) = 1.202\,, \qquad \zeta(4) = \pi^4/90$$

Number densities - Solutions

- MB, small T: this coincides with FD and BE when $E \simeq m \gg T$ (non relativistic)

$$e^{-E/T} \rightarrow e^{-m/T-p^2/(2mT)}$$

$$\text{define } x \equiv p/\sqrt{2mT} \rightarrow n = \frac{e^{-m/T}}{\pi^2}\,[2mT]^{3/2} \underbrace{\int_0^\infty dx\, x^2\, e^{-x^2}}_{\sqrt{\pi}/2}$$

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- FD/BE, large T: use again $x \equiv p/T$, so $n = \frac{T^3}{\pi^2} \int_0^\infty dx\, \frac{x^2}{e^x \pm 1}$

Now you can use the $\zeta(x)$ definitions, so the integral is:

BE$\to$ $\zeta(3)\Gamma(3) = 2\zeta(3)$ FD$\to$ $3/4\zeta(3)\Gamma(3) = 3\zeta(3)/2$

Energy densities

1b

Repeat the previous exercise, but now compute

$$\rho = g \int \frac{d^3p}{(2\pi)^3} E\, f(p)$$

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- (non relativistic), that is MB, FD or BE, small T ($E \simeq m \gg T$):

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- FD/BE, large T, $E \simeq p$: $x \equiv p/T$, so $\rho = \frac{T^4}{\pi^2} \int_0^\infty dx\, \frac{x^3}{e^x \pm 1}$

As done for computing n, use the $\zeta(n)$ definition:

BE$\rightarrow$ $\zeta(4)\Gamma(4) = \pi^4/15$ FD$\rightarrow$ $7/8\zeta(4)\Gamma(4) = 7\pi^4/120$

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where $\rho_{crit} \simeq 1.88h^2 \times 10^{-32}$ kg cm^{-3} and $m_p \simeq 1.7 \times 10^{-27}$ kg

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normalize: $\eta_b \simeq 5.9 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.022}\right)$

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