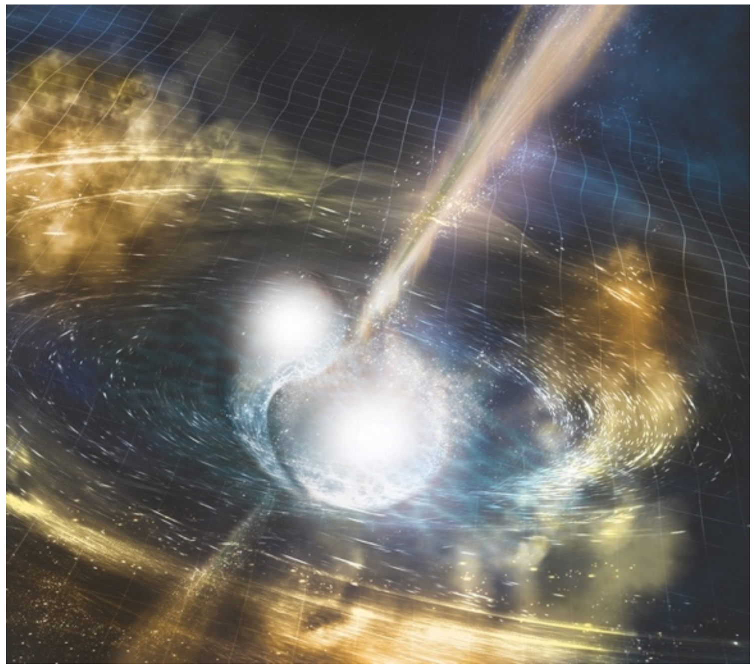
# Gravitational waves: the new cosmic messengers



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# Outline

First talk:

- Basics of gravitational waves
- Sources of gravitational waves
- Strong-field gravity and numerical relativity
- Detection challenge

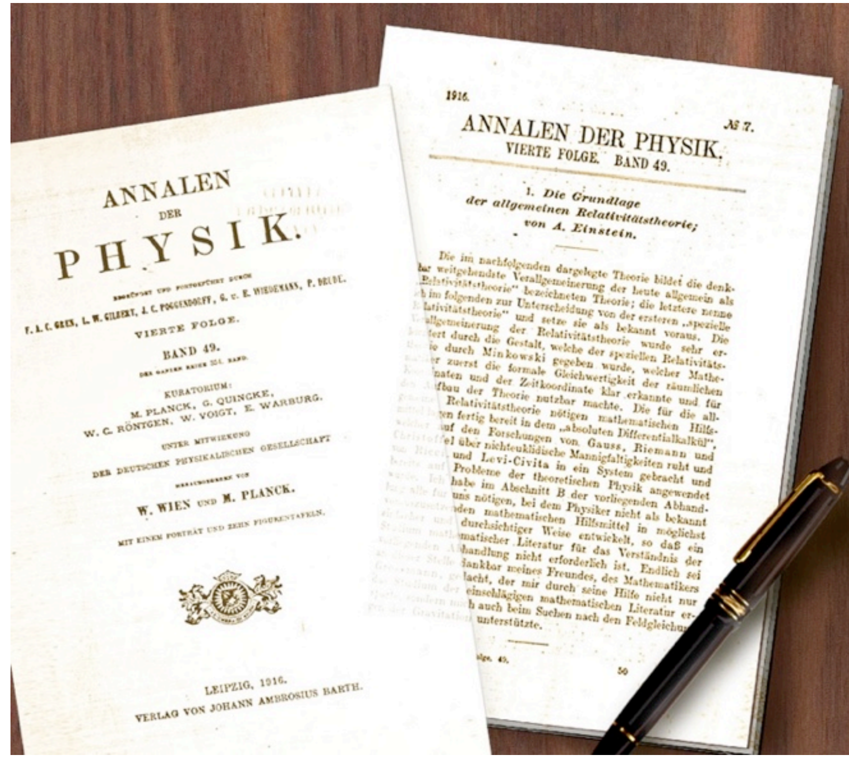
Second talk:

- Advanced LIGO/Virgo detections

The existence of gravitational radiation was predicted by Albert Einstein about 100 years ago ...

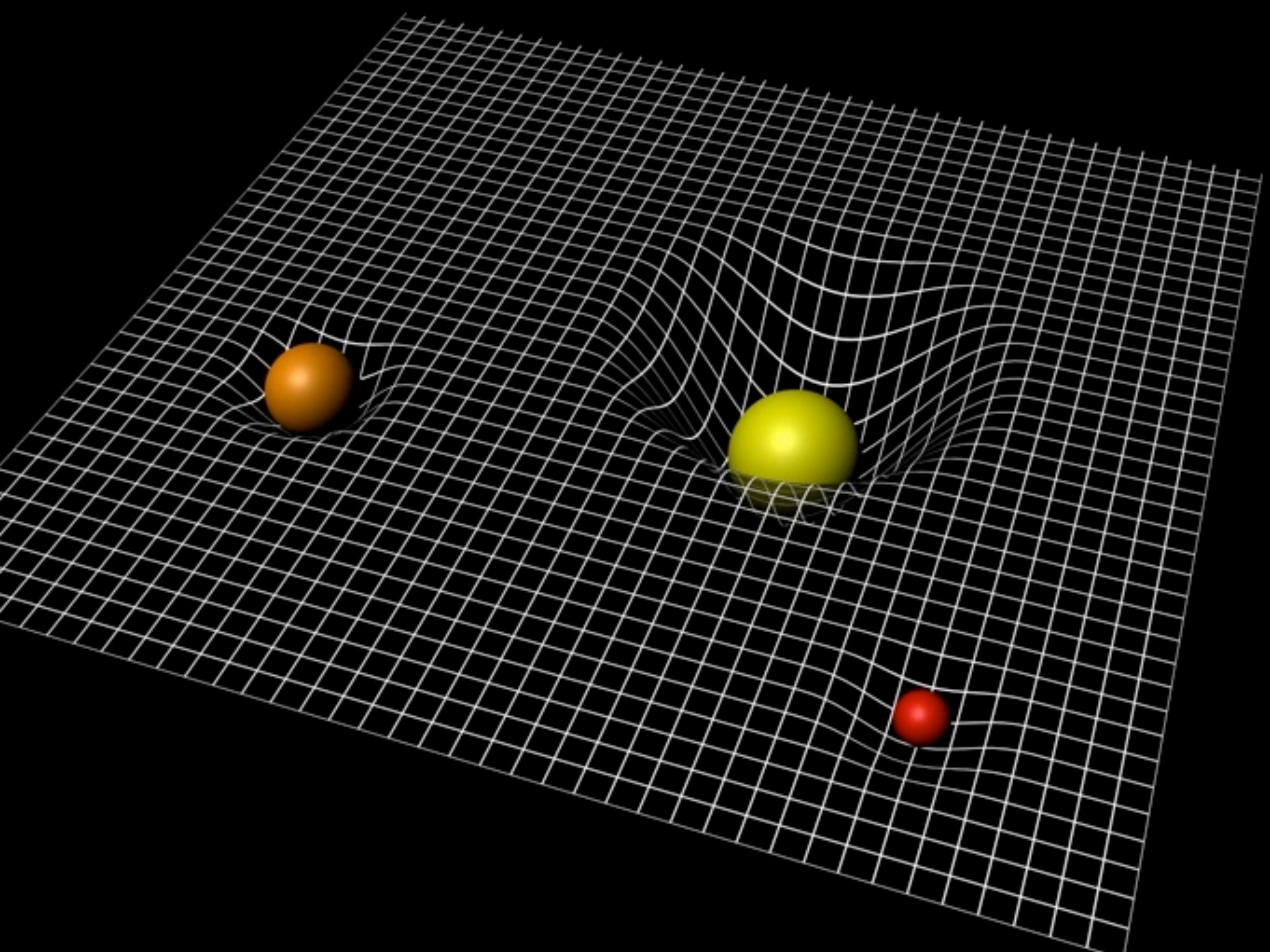
There were hopes to have a direct detection towards the theory's centennial.

Centennial Day:
November 25, 2015




Convincing observational evidence obtained 60 years after the prediction (Binary Pulsar).

... but we had to wait until September 14, 2015 to accomplish the direct confirmation of its existence



# Weak gravitational fields

The exterior gravitational field from a spherical body or a slowly-rotating body as the Earth or the Sun can be approximated by the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

On the surface, we can compute the compactness parameter:

$$\frac{GM}{Rc^2} = \begin{cases} 2 \times 10^{-6} & \text{for the Sun} \\ 7 \times 10^{-10} & \text{for the Earth} \end{cases}$$

Since $\frac{M}{R} \ll 1$ the metric is very close to Minkowski (flat spacetime):

$$g_{tt} = -\left(1 - \frac{2GM}{rc^2}\right) \sim -1 \qquad g_{rr} = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \sim 1 + \frac{2GM}{rc^2} \sim 1$$

Only for compact objects, like neutron stars or black holes, $\frac{M}{R} \sim 1$

For most of the universe, away from compact objects, the gravitational field can be regarded as weak.

For weak gravitational fields we can use approximations for the spacetime metric, that greatly simplify Einstein's equations:

**Vacuum**: no $M/R$ terms

**Newtonian gravity**: $M/R$ terms in the dynamics of the system

**Post-Newtonian gravity of order N:** $(M/R)^N$ terms in the dynamics

## Weak gravitational field: linearized Einstein equations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;\; \text{with} \;\; |h_{\mu\nu}| \ll 1$$

Indices can be raised and lowered using Minkoski's metric.

Einstein's equations $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

for a weak gravitational field reduce to

$$\boxed{h_{\mu\nu,\alpha}{}^{\alpha} + h_{,\mu\nu} - h_{\mu\alpha,\nu}{}^{\alpha} - h_{\nu\alpha,\mu}{}^{\alpha} + \eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} + h_{,\beta}{}^{\beta}) = -16\pi T_{\mu\nu}}$$

- Second-order PDEs.
- D'Alembertian operator acting on $h_{\mu\nu}$ (hint of a wave equation!)
- Source terms on the right-hand side.

The previous equation is reminiscent of a wave equation, but with additional terms. To cast it into a wave equation we have to follow two steps:

- Redefine variables ("bar" operator)
- Use the gauge freedom associated with the covariance of EE.

Bar operator (acting on a symmetric tensor):

$$\bar{\mathcal{T}}_{\mu\nu} \equiv \mathcal{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{T} \qquad\qquad \mathcal{T} \equiv \eta^{\alpha\beta}\mathcal{T}_{\alpha\beta}$$

Applying this operation to the linearized equations, we obtain:

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \bar{h}_{\mu\alpha,\nu}{}^{\alpha} - \bar{h}_{\nu\alpha,\mu}{}^{\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{\alpha\beta} = -16\pi T_{\mu\nu}$$

The bar operator simplifies Einstein's equations by absorbing the trace of *h* into the definition of $\bar{h}_{\mu\nu}$

Further simplification of the equations is achieved by using the Lorenz gauge (as in EM):

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \Rightarrow \boxed{\Box\bar{h}_{\mu\nu} \equiv \bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -16\pi T_{\mu\nu}}$$

Explicit wave equation for $\bar{h}_{\mu\nu}$

The linearized equations are invariant under infinitesimal coordinate transformations:

$$x^{\mu'} = x^{\mu} + \xi^{\mu}, \quad |\xi^{\mu}| \ll 1$$

Applying this transformation to the metric

$$g_{\mu'\nu'} = g_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \mathcal{O}(\xi^2)$$

If $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1)$ then $g_{\mu'\nu'} = \eta_{\mu'\nu'} + \mathcal{O}(1)$

Therefore, infinitesimal coordinate transformations do not change the weak character of the field, i.e. if $\left|h_{\mu\nu}\right| \ll 1$ then $\left|h_{\mu'\nu'}\right| \ll 1$

This means there is extra gauge freedom in the linearized system that we can fix arbitrarily. Since we have already made a choice associated with general covariance (Lorenz gauge), the gauge freedom associated with infinitesimal coordinate transformation fixes the possible values of $\xi_{\mu}$

$$\xi^{\alpha,\beta}{}_{\beta} = \Box\xi^{\alpha} = 0$$

Degrees of freedom:

10 components of a symmetric tensor of rank 2
-4 gauge conditions associated with general covariance (Lorenz gauge)
-4 gauge conditions assoaciated with infinitesimal coordinate transformations

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2 degrees of freedom

Therefore, linearized Einstein's equations have **two physical degrees of freedom**. As we shall see, they are associated with the two polarizations of gravitational radiation.

**Vacuum, plane-wave solutions:**

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \;\Rightarrow\; \bar{h}_{\mu\nu} = \mathrm{Re}[A_{\mu\nu}\, e^{ik_\alpha x^\alpha}]$$

satisfying the following conditions: $k_\alpha k^\alpha = 0\,, \quad A_{\mu\alpha} k^\alpha = 0$

The wave vector $k_\mu$ gives information on the wave frequency $\omega$ and on the direction of propagation $n^i$

$$\omega \equiv k^0 = \sqrt{k^i k_i}\,, \quad n^i \equiv \frac{k^i}{k^0}$$

Since we can fix the gauge freedom under infinitesimal transformations, we have the freedom to impose 4 additional conditions. It is customary to use the so-called *gauge TT*

$$A_{\mu\nu}u^\mu = 0 \qquad\qquad u^\mu \text{ observer's 4-velocity}$$

$$A_\mu{}^\mu = 0$$

Let us consider a 3+1 spacetime foliation with coordinates $x^\mu = (t, x^i)$

Let us choose an observer with $u^\mu = (1,0,0,0)$ (Lorentz frame)

Let $h^{\mathrm{TT}}_{\mu\nu}$ be the value of $\bar{h}_{\mu\nu}$ in the TT gauge. This tensor has the following properties:

$$h^{\mathrm{TT}}_{\mu 0} = 0 \qquad \text{transverse}$$

$$h^{\mathrm{TT}}_{\mu j}{}^{,j} = 0 \qquad \text{divergence free}$$

$$\eta^{ij}h^{\mathrm{TT}}_{ij} = 0 \qquad \text{traceless}$$

$$h^{\mathrm{TT}}_{ij}n^i = 0 \qquad \text{transverse to the direction of propagation}$$

$h^{\mathrm{TT}}_{\mu\nu}$ is also traceless: $h^{\mathrm{TT}} \equiv \eta^{\mu\nu}h^{\mathrm{TT}}_{\mu\nu} = \eta^{00}h^{\mathrm{TT}}_{00} + \eta^{ij}h^{\mathrm{TT}}_{ij} = 0$

Therefore, $\bar{h}_{\mu\nu}^{\mathrm{TT}} = h_{\mu\nu}^{\mathrm{TT}}$ and we can remove the bar notation. Einstein's equations in vacuum can thus be written as $\Box h_{ij}^{\mathrm{TT}} = 0$

We can list the following properties of gravitational waves:

- They propagate at the speed of light.
- We need a tensor of rank 2 to describe them.
- They are invariant under infinitesimal transformations.
- They have two physical degrees of freedom (polarizations).
- They are transverse to the direction of propagation.

If we quantized this field, as a consequence of the first three properties, the resulting particle - the graviton - would have the following properties:

- zero mass
- spin 2 (boson)
- a gauge boson (carrier of the gravitational force)

**Polarization:**

Let us consider Cartesian coordinates and a travelling gravitational wave in the *z* direction, i.e. $k_x = k_y = 0$ . The wave-plane solution reads:

$$h_{\mu\nu}^{\mathrm{TT}} = \mathrm{Re}[A_{\mu\nu} e^{-i\omega(t-z)}] \qquad \omega = k^0 = k_z$$

In this case, the three properties of the gauge TT imply:

$$h_{00}^{\mathrm{TT}} = h_{i0}^{\mathrm{TT}} = 0$$
$$h_{iz,z}^{\mathrm{TT}} = i\omega h_{iz}^{\mathrm{TT}} = 0$$
$$h_{xx}^{\mathrm{TT}} + h_{yy}^{\mathrm{TT}} = 0$$

Therefore, there are only two degrees of freedom left, corresponding to the two possible polarizations:

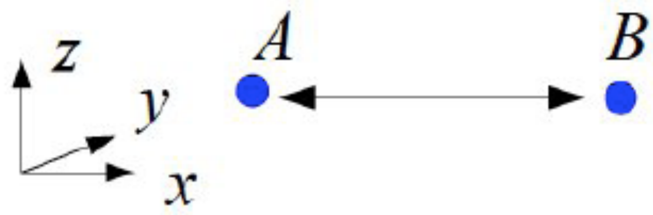
$$\boxed{\begin{aligned} h_{xz}^{\mathrm{TT}} = h_{yz}^{\mathrm{TT}} = h_{zz}^{\mathrm{TT}} &= 0 \\ h_{xx}^{\mathrm{TT}} = -h_{yy}^{\mathrm{TT}} &= \mathrm{Re}[A_+ e^{-i\omega(t-z)}] \quad \text{(+ polarization)} \\ h_{xy}^{\mathrm{TT}} = h_{yx}^{\mathrm{TT}} &= \mathrm{Re}[A_x e^{-i\omega(t-z)}] \quad \text{(x polarization)} \end{aligned}}$$

Written in matrix form:

$$h_{ij}^{\mathrm{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx}^{\mathrm{TT}} & h_{xy}^{\mathrm{TT}} & 0 \\ 0 & h_{xy}^{\mathrm{TT}} & -h_{xx}^{\mathrm{TT}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Re}\left\{ \left[ A_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A_x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] e^{-i\omega(t-z)} \right\}$$

# Physical interpretation of the two polarizations

Let us consider the effect of a plane wave travelling in the *z* direction on two test particles (A and B) placed on the *x* axis and initially separated (before the gravitational wave arrives) by a distance $\Delta l_0$ .




The metric in the gauge TT reads: $ds^2 = -dt^2 + (\eta_{ij} + h_{ij}^{\mathrm{TT}})dx^i\, dx^j$

Since A and B are on the x-axis, $dy = dz = 0$ then, for a given time $(dt = 0)$ the metric reads:

$$ds^2 = dl^2 = (\eta_{xx} + h_{xx}^{\mathrm{TT}})dx^2$$

Therefore, the distance between the two test particles is given by:

$$\Delta l = \int_A^B \sqrt{ds^2} = \int_A^B \sqrt{1 + h_{xx}^{\mathrm{TT}}} \approx \int_A^B \left(1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}\right) dx \approx \Delta l_0 \left(1 + \frac{1}{2} h_{xx}^{\mathrm{TT}}\right)$$

with

$$\Delta l_0 = \int_A^B dx$$

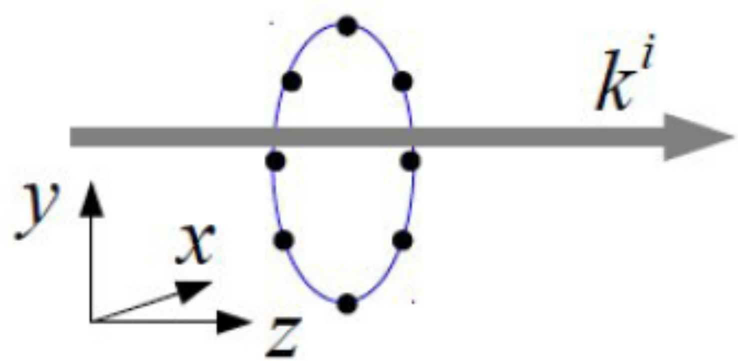
Measuring the variation of the relative distance between test particles is at the core of the design of current gravitational-wave detectors.

In general, the position of test particle B seen from the position of test particle A is given by (see e.g. section 35.5 of Misner et al 1973):

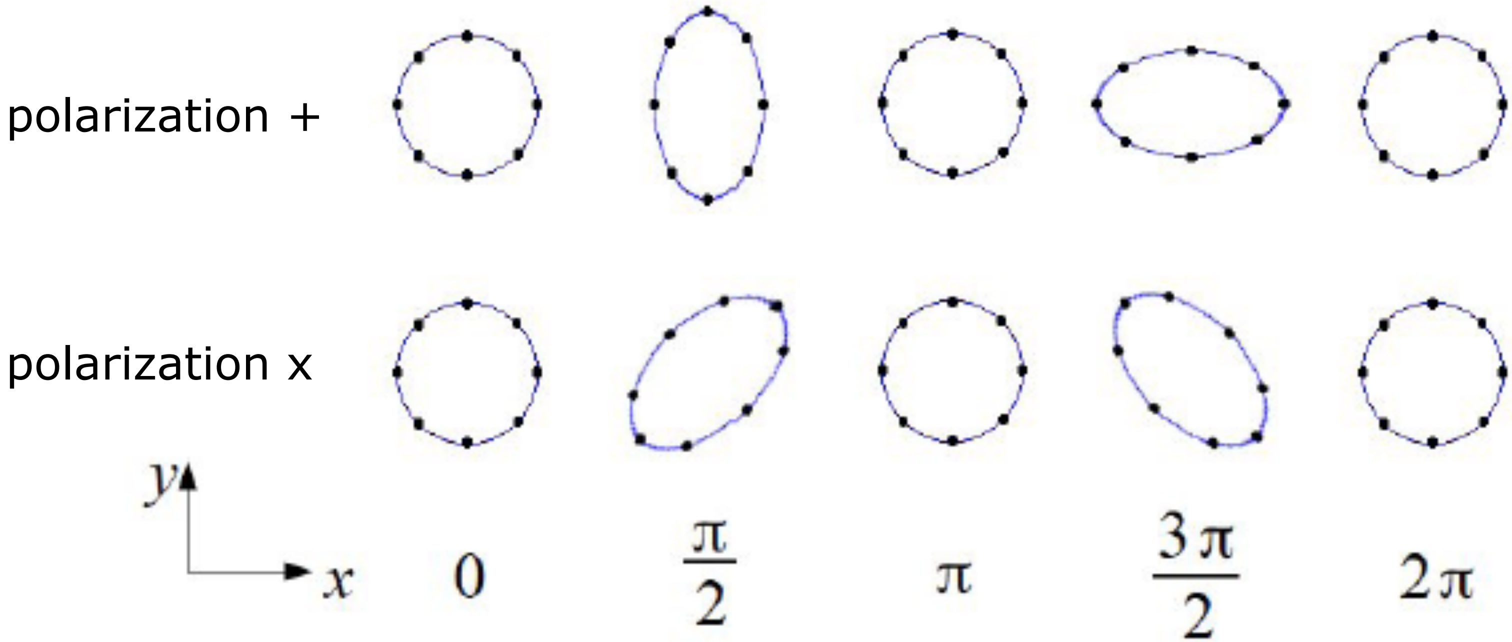
$$x_B^j = x_B^{j(0)} + \mathrm{Re}\left[\frac{1}{2} A_0 e^{-i\omega(t-z)} x_B^{j(0)}\right] = x_{B(0)}^k \left(\delta_{jk} + \frac{1}{2} h_{jk}^{\mathrm{TT}}\right)$$

where $x_B^{j(0)}$ is the initial position of particle B before the gravitational wave arrives.

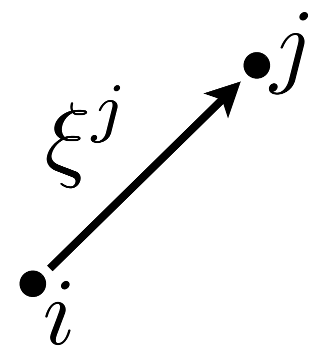
We can apply this expression to a circular set of test particles that are hit by a gravitational wave with wave vector $k^i$ perpendicular to the plane of the circle.




The particles will oscillate around their equilibrium positions with an oscillation frequency $\omega$ . The form of the oscillation pattern will depend on the polarization of the gravitational wave. The patterns created by the + polarization and the x polarization are rotated by $\pi/4$ .




Two test particles will experience a **tidal acceleration**:

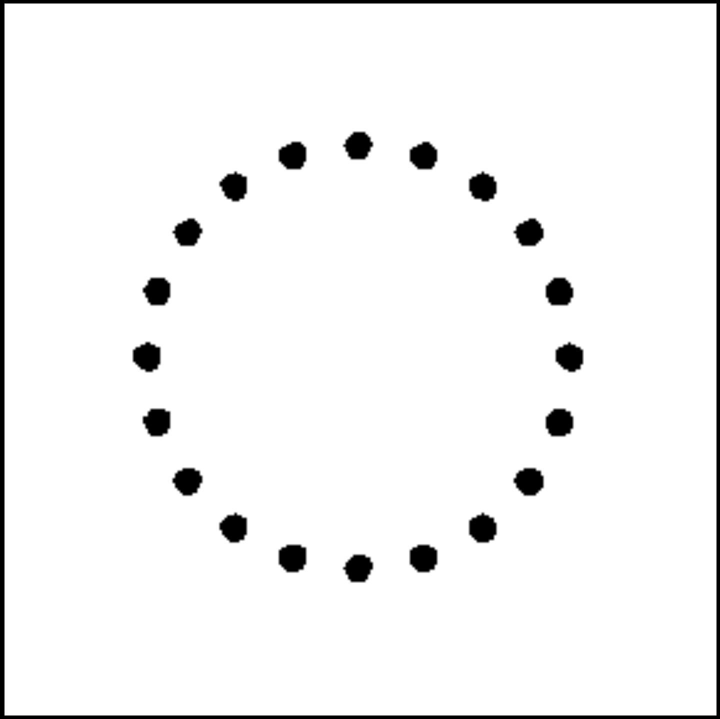


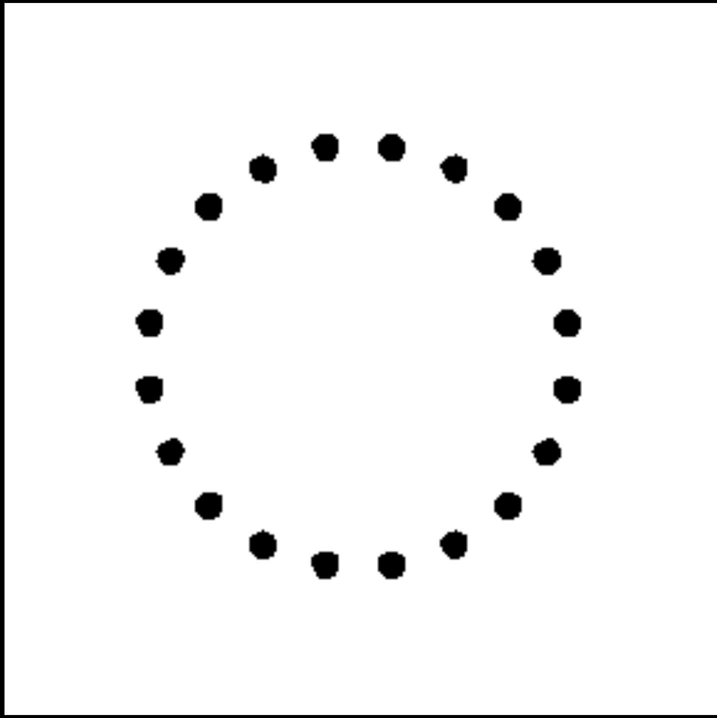

$$\ddot{\xi}_j = \frac{D^2 \xi_j}{D\tau^2} = -R_{j0k0}\,\xi_k = \frac{1}{2}\frac{\partial^2 h_{jk}}{\partial t^2}\xi_k = \mathrm{Re}\left[-\frac{1}{2}\omega^2 A_0 e^{-i\omega(t-z)}\xi_k\right]$$

# Tidal field

A gravitational wave exerts tidal forces on to any massive object it goes through.

Distances between test masses expand and contract.





$$h^{\mathrm{TT}}_{jk} = h_+\, e^+_{jk} + h_\mathrm{x}\, e^\mathrm{x}_{jk}$$

traceless, transverse
symmetric tensor

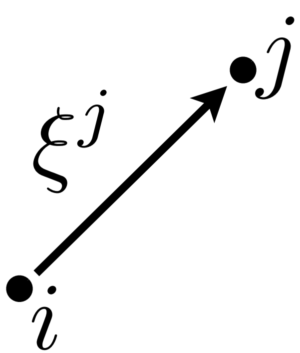
$$e^+_{jk}\,,\, e^\mathrm{x}_{jk}$$

polarization tensors

$h^{\mathrm{TT}}_{jk}$ is equivalent to the vector potential of electromagnetism in the Lorenz gauge $\bar{h}_{\mu\nu,\nu} = 0$

$$\boxed{\begin{array}{c} A_0 = 0, \quad A_{i,i} = 0, \quad \Box A_i = 0 \\ h^{\mathrm{TT}}_{0\mu} = 0, \quad h^{\mathrm{TT}}_{jk,k} = 0, \quad \Box h^{\mathrm{TT}}_{jk} = 0 \end{array}}$$

A GW interacting with a test particle exerts a transverse acceleration.




$$\ddot{\xi}^j = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{jk}\,\xi^k \qquad \delta\xi^j = \frac{1}{2}h^{\mathrm{TT}}_{jk}\,\xi^k$$

relative deformation $\dfrac{\delta\xi}{\xi} \sim h$

# Einstein Equations

**GEOMETRY** **MATTER**

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

Summary (so far): Far from the sources of the gravitational field, where the field is weak, the Einstein equations turn into a **wave equation**.

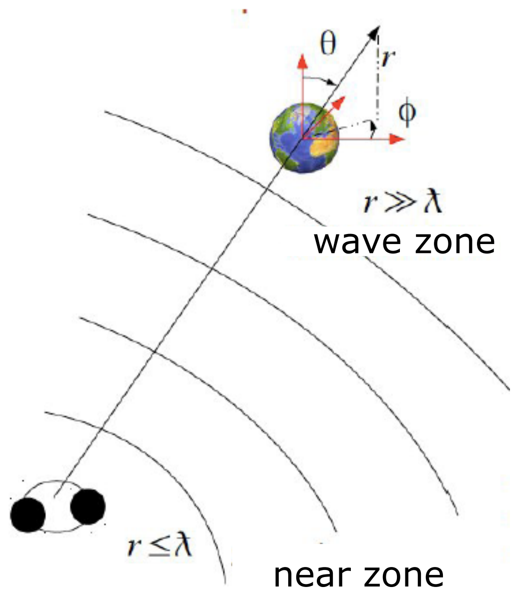
Weak field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (Minkowski + perturbation)

$$\Box\, \bar{h}_{\mu\nu} := \left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}\, h$$

# Multipolar solutions

Let us consider a source of gravitational waves of compact support (i.e. confined in a region of space of size R). Far from the source the spacetime is Minkowski (weak): the gravitational waves emitted by the source will be solution of the linearized Einstein's equations.




We search for solutions valid in the **wave zone** where linearization is possible. The solution will be given as a function of the distance to the source and direction of observation.

The wave equation for a 3-tensor of rank 2 $\mathcal{T}_{ij}$

$$-\partial_{tt}\mathcal{T}_{ij} + \Delta\mathcal{T}_{ij} = 0$$

admits solutions of the form (Thorne 1980):

$$\Phi_{ij}^{\pm\omega\lambda l' lm}(t,r,\theta,\varphi) = \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \left(j_{l'}(\omega r) \pm i y_{l'}(\omega r)\right) T_{ij}^{\lambda l',lm}(\theta,\varphi)$$

$j_l(z), y_l(z)$ spherical Bessel functions (first and second kind)

$T_{ij}^{\lambda l',lm}(\theta,\phi)$ pure-orbital tensor spherical harmonics $\pm$ outgoing/incoming

$$\lambda = 0 \quad \rightarrow \quad l' = l$$

$$\lambda = 2 \quad \rightarrow \quad l' = l, l \pm 1, l \pm 2$$

$\longrightarrow$ 6 possible tensor spherical harmonics

It is more convenient to use *pure-spin tensor spherical harmonics* which are linear combinations of the pure-orbital harmonics. There are 6 types:

$T_{ij}^{L0,lm}(\theta,\phi), \quad T_{ij}^{T0,lm}(\theta,\phi)$ spin-0, longitudinal/transverse

$T_{ij}^{E1,lm}(\theta,\phi), \quad T_{ij}^{B1,lm}(\theta,\phi)$ spin-1, electric/magnetic

$T_{ij}^{E2,lm}(\theta,\phi), \quad T_{ij}^{B2,lm}(\theta,\phi)$ spin-2, electric/magnetic

The spin of the field indicates the rank of the tensor needed to build each of the tensor harmonics. The concept of \`electric' and \`magnetic' is the generalization to tensors of the concept of polar vector and axial vector. The form of the pure-spin harmonics is (w.r.t. an orthonormal basis)

$$T_{ij}^{L0,lm}(\theta,\phi) \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_{ij}^{T0,lm}(\theta,\phi) \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{ij}^{E1,lm}(\theta,\phi), T_{ij}^{B1,lm}(\theta,\phi) \propto \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \qquad T_{ij}^{E2,lm}(\theta,\phi), T_{ij}^{B2,lm}(\theta,\phi) \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & -a \end{pmatrix}$$

Due to the properties of $h_{ij}^{\mathrm{TT}}$, in particular being traceless and transverse to the direction of propagation, the only tensor spherical harmonics that appear in the solution are E2 and B2, which are precisely those representing a spin-2 field.

Thus, the solution of the linearized Einstein equations in the TT gauge reads:

$$h_{ij}^{\mathrm{TT}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{\infty} d\omega \, \Big( a^{+}_{E2,lm}(\omega) \Phi_{ij}^{+\omega E2,lm} + a^{-}_{E2,lm}(\omega) \Phi_{ij}^{-\omega E2,lm} + a^{+}_{B2,lm}(\omega) \Phi_{ij}^{+\omega B2,lm} + a^{-}_{B2,lm}(\omega) \Phi_{ij}^{-\omega B2,lm} \Big)$$

In the wave zone, integrating in $\omega$ and considering only the outgoing-wave solution, the general solution reads:

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ I^{lm}(t-r) T_{ij}^{E2,lm} + S^{lm}(t-r) T_{ij}^{B2,lm} \right]}$$

$I^{lm}(t-r)$ mass-multipole moments

$S^{lm}(t-r)$ current-multipole moments

**Properties of the multipolar expansion of the gravitational wave:**

- Amplitude falls-off like $1/r$
- The emission is not spherically symmetric; it depends on the form of the tensor spherical harmonics.
- The mass and current multipoles contain all the information about the gravitational radiation of the system.
- The information transmitted by the wave propagates at the speed of light (it can be written in terms of the retarded time).
- The two polarizations can be expressed in terms of the non-zero components of $h_{ij}^{\mathrm{TT}}$

$$h_+ = h_{\theta\theta}^{\mathrm{TT}} = -h_{\varphi\varphi}^{\mathrm{TT}}$$

$$h_{\mathrm{x}} = h_{\theta\varphi}^{\mathrm{TT}} = h_{\varphi\theta}^{\mathrm{TT}}$$

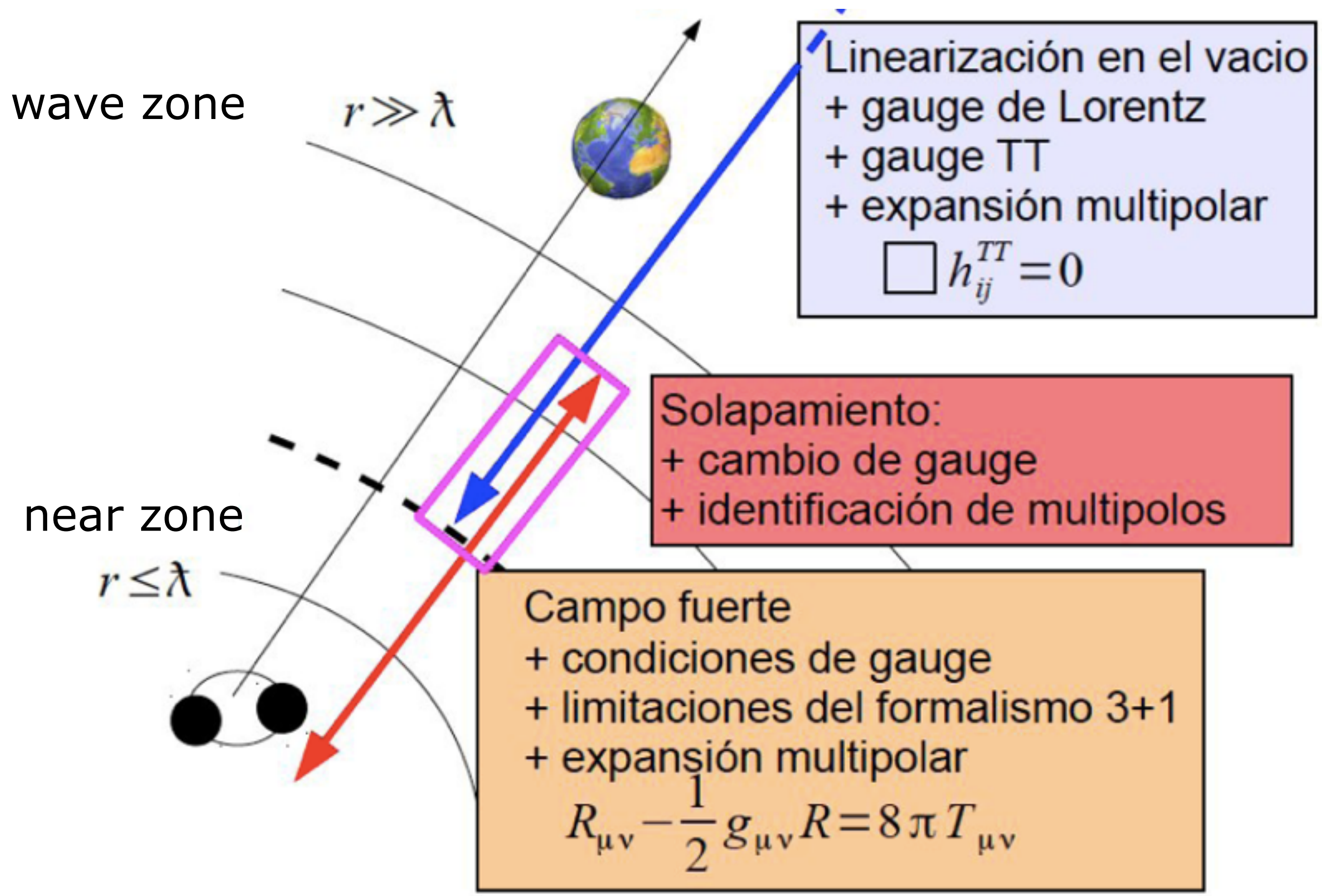
For example, if we consider a gravitational wave whose only contribution is from the **mass quadrupole**, $I^{20}$ , then

$$h_+ = \frac{1}{r} I^{20}(t-r)\frac{1}{8}\sqrt{\frac{15}{\pi}}\sin^2\theta$$

$$h_{\mathrm{x}} = 0$$

# Generation of gravitational waves

We need to relate the strong-gravity near zone where the waves are generated (numerical relativity solution) with the weak-gravity, wave zone where the previous solution is valid. We will need to match the inner solution with the outer one, to obtain the multipoles (change of gauge).



# Post-Newtonian sources: the quadrupole formula

Let us come back to the perturbation of the Minkowski metric:

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

If we keep the terms non linear in the perturbation, the resulting Einstein's equations read:

$$\bar{h}^{\mu\nu}{}_{,\alpha\beta}\eta^{\alpha\beta} = -16\pi(T^{\mu\nu} + t^{\mu\nu})$$

$t^{\mu\nu}$ is a stress-energy "pseudotensor" which contains the metric terms of the equations non-linear in $h_{\mu\nu}$ . It may be large close to the sources and cannot be neglected there.

Since the previous equation is a wave equation, it admits `ongoing' wave solutions in terms of Green integrals:

$$\bar{h}^{\mu\nu}(t, \mathbf{x}) = 4\int_{\mathrm{space}} \frac{T^{\mu\nu}(t', \mathbf{x}') + t^{\mu\nu}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}'$$

$$t' \equiv t - |\mathbf{x} - \mathbf{x}'|$$

In general this integral cannot be computed since $t^{\mu\nu}$ is a function of $h_{\mu\nu}$. The integral can however be computed in the so-called **small velocity approximation**.

Let us consider a source of gravitational waves of size *R* and mass *M* whose dynamical timescale is *T*. Let us also consider that the velocities involved are small

$$V \sim \frac{R}{T} \ll 1$$

The wavelength of the emitted waves is $\lambdabar \sim cT$ and thus $\boxed{R \ll \lambdabar}$

The situation in which this condition is valid can be related with the mass of the system using the <u>virial theorem</u>:

$$U_{\rm grav} \sim E_{\rm kin} \;\rightarrow\; G\frac{Mm}{R} \sim \frac{mR^2}{T^2} \;\rightarrow\; \frac{1}{T^2}\frac{GM}{R^3} \;\rightarrow\; \lambdabar \sim cT \sim R\left(\frac{GM}{Rc^2}\right)^{-1/2}$$

Therefore, the approximation of small velocity is equivalent to:

$$\boxed{\frac{GM}{Rc^2} \ll 1}$$

This approximation is this valid for objects which are **not too compact**, for which Newtonian dynamics is valid.

In this approximation, $t^{\mu\nu}$ falls with *r′* "sufficiently fast" close to the source, *r′~R,* so that $T^{\mu\nu} + t^{\mu\nu}$ only contributes to the Green integral for *r′~R*.

To compute the integral at distances $r \gg R$ we can use the fact that $r \gg r'$ and therefore:

$$|\mathbf{x}| \equiv r \gg |\mathbf{x}'| \equiv r' \;\rightarrow\; |\mathbf{x} - \mathbf{x}'| = r + \mathcal{O}(r^2)\ ,\ t' = t - r + \mathcal{O}(r^2)$$

In this limit the Green integral reads:

$$\bar{h}^{\mu\nu}(t, \mathbf{x}) = \frac{4}{r} \int_{r' \sim R} [T^{\mu\nu}(t - r, \mathbf{x}') + t^{\mu\nu}(t - r, \mathbf{x}')]\, d^3\mathbf{x}'\ ,\ r = |\mathbf{x}| \gg R$$

This integral can be computed because $t^{\mu\nu}$ is a function of $h_{\mu\nu}$ in *r'~R* while the resulting $h_{\mu\nu}$ is in $r \gg \lambda \gg R$

Using the definition of the Lorenz gauge and Einstein's equation, it is possible to obtain a Bianchi identity for the entire stress-energy tensor (including the gravitational field)

$$(T^{\mu\nu} + t^{\mu\nu})_{,\nu} = 0$$

We can use this equation to obtain the components of $\bar{h}^{ij}$ in terms of the 00 components of the stress-energy tensors:

$$\bar{h}^{ij} = \frac{2}{r} \frac{d^2}{dt^2} \int [T^{00} + t^{00}]\, x'^i x'^j\, d^3\mathbf{x}' + \mathcal{O}(1/r^2)$$

Since in the small-velocity approximation the dynamics is Newtonian, we can approximate

$$T^{00} + t^{00} = \rho + \mathcal{O}(M/R)$$

where $\rho$ is the rest-mass density. This way, in the TT gauge

$$h_{ij}^{\rm TT} = \frac{2}{r}\frac{G}{c^4}\frac{d^2 Q_{ij}(t - r/c))}{dt^2} + \mathcal{O}(1/r^2)$$

**quadrupole formula**

$$Q_{ij} = \int \rho \left( x^i x^j - \frac{1}{3}\delta^{ij} r^2 \right) d^3\mathbf{x} + \mathcal{O}(M/R)$$

**quadrupole moment or quadrupole**

This formula allows to compute the gravitational radiation emitted by a source by doing simple integrals over the density distribution of the source. The error is about 10% for objects like neutron stars where $M/R \sim 0.1$

We can compute the components of the tensor in spherical coordinates for an axisymmetric matter distribution $\rho(t, r, \theta)$

$$h_+ = h_{\theta\theta}^{\rm TT} = -h_{\varphi\varphi}^{\rm TT} = \frac{2}{r}\sin^2\theta \frac{d^2}{dt^2}\int \rho \left( \frac{3}{2}\cos^2\theta - \frac{1}{2} \right) r^2 d^3\mathbf{x} + \mathcal{O}(M/R) + \mathcal{O}(1/r^2)$$

$$h_{\rm x} = h_{\theta\varphi}^{\rm TT} = h_{\varphi\theta}^{\rm TT} = 0$$

This expression can be compared with the multipolar solution obtained previously to notice that, in this approximation, there is only contribution to the term $I^{20}$

$$h_+ = \frac{1}{r} I^{20}(t-r) \frac{1}{8}\sqrt{\frac{15}{\pi}} \sin^2\theta + \mathcal{O}(1/r^2)$$

$$I^{20} = \frac{d^2}{dt^2} \int \rho \left( \frac{3}{2}\cos^2\theta - \frac{1}{2} \right) r^2 d^3\mathbf{x} + \mathcal{O}(M/R)$$

We can consider the following particular cases:

- Spherically-symmetric matter distribution $\rho(t,r)$

$$Q \propto \int_0^\infty \rho(t,r) r^4 dr \int_0^\pi \left[ \frac{3}{2}\cos^2\theta - \frac{1}{2} \right] \sin\theta \, d\theta = 0$$

There is no gravitational radiation. An example could be the gravitational collapse of a spherical star.

- Non-spherical matter distribution with no time dependence $\rho(r,\theta,\varphi)$

$$Q \neq 0 \quad \text{but} \quad \frac{dQ}{dt} = 0$$

There is no gravitational radiation. An example could be a rotating star. It could be very deformed (oblate shape) but it would not radiate.

- Non-spherical matter distribution travelling with uniform speed $\rho(r, \theta, \varphi)$

$$\frac{\partial^2 \rho}{\partial t^2} = 0 \quad \rightarrow \quad \frac{d^2 Q}{dt^2} = 0$$

There is no gravitational radiation. An example could be a star travelling at constant speed across space. It is also a consequence of Lorentz invariance.

- Non-spherical matter distribution undergoing accelerations $\rho(r, \theta, \varphi)$

$$\frac{\partial^2 \rho}{\partial t^2} \neq 0 \quad \longrightarrow \quad h_+ \propto \frac{d^2 Q}{dt^2} \neq 0$$

There *is* gravitational radiation. Examples: coalescing compact binaries, generic gravitational collapse of the core of massive stars.

---

The quadrupole formula can also be used to estimate the amplitude, frequency, and wavelength of the gravitational waves emitted by a source. Using dimensional analysis

$$h_+ \sim \frac{1}{r} \frac{MGV^2}{c^4} = \frac{1}{r} \frac{MGR^2}{c^4 T^2}$$

Applying the expressions obtained with the virial theorem, we can express the quantities in terms of *M* and *R*:

$$\boxed{h_+ \sim \frac{1}{r}\frac{M^2G^2}{Rc^4} \;\; , \;\; \omega = \frac{c}{\lambda} \sim \frac{1}{T} \sim \sqrt{\frac{GM}{R^3}} \;\; \text{if} \;\; \frac{GM}{Rc^2} \ll 1}$$

The <u>gravitational-wave luminosity,</u> the energy emitted in gravitational waves per unit of time, is defined as

$$L_{\mathrm{GW}} = \frac{dE_{\mathrm{GW}}}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3} \right\rangle$$

where $\langle \; \rangle$ indicates average on several periods of evolution of the source, as the energy of the radiation cannot be localised in a single wavelength.

In terms of *M* and *R*:

$$L_{\mathrm{GW}} \sim \frac{c^5}{G}\left(\frac{GM}{c^2R}\right)^5$$

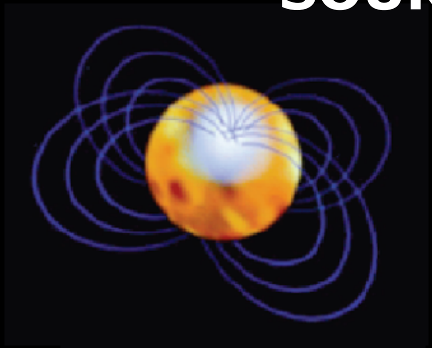
We might apply this expression for the case of a black hole (only to get a rough estimate, as this case is out of the scope of applicability):

$$L_{\mathrm{GW}} \sim \frac{c^5}{G} \approx 3.6 \times 10^{59}\,\mathrm{erg\,s^{-1}}$$

$$L_{\mathrm{EM}}^{\mathrm{sun}} \approx 10^{33}\,\mathrm{erg\,s^{-1}}$$

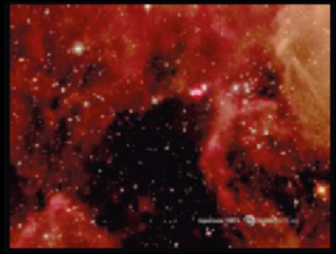
$$L_{\mathrm{EM}}^{\mathrm{supernova}} \approx 10^{43}\,\mathrm{erg\,s^{-1}}$$

# SOURCES OF GRAVITATIONAL WAVES

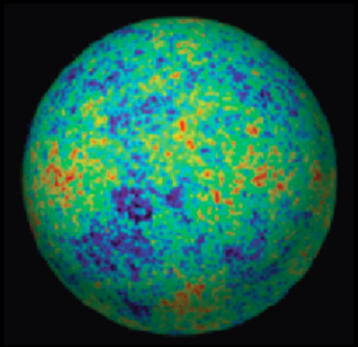


**Continuous sources:** rotating neutron stars and pulsars. Periodic signal.

**Merger of compact binaries:** inspiral, merger and pulsations of black holes and neutron stars. Chirp signal.

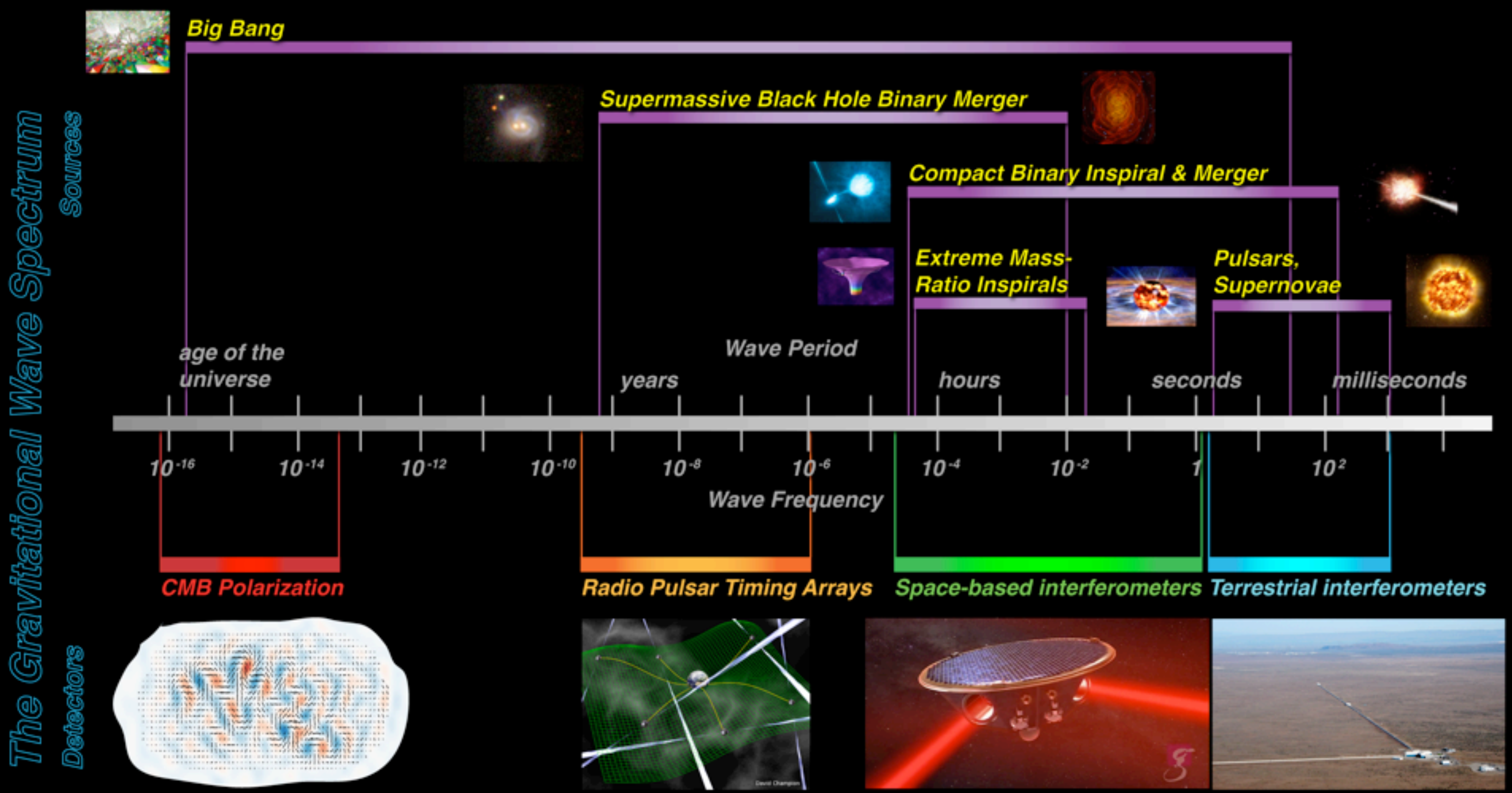


**Brief events:** supernovae, GRBs, unmodelled transient sources. Burst signal.



**Stochastic sources:** gravitational wave background from the Big Bang.

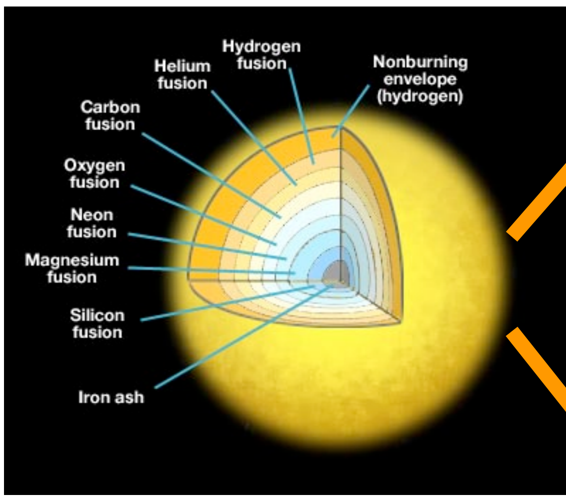
# THE GRAVITATIONAL-WAVE SPECTRUM



# Gravitational collapse of massive stars

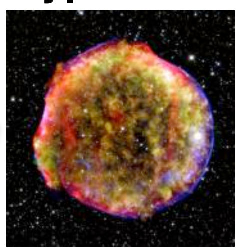
- among the most important sources of GWs (M>8 Msun)
- complex waveforms: key info about scenario, NS physics and EOS, specially with observations of EM emission and neutrino emission
- could be mechanism behind **long** Gamma Ray Bursts

## Fate of massive stars (8-100 $M_{sun}$)

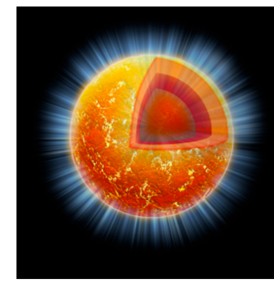



**Collapse of iron core**

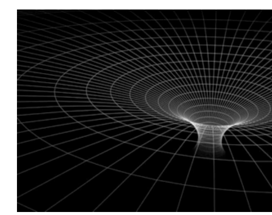
**Type II SN**



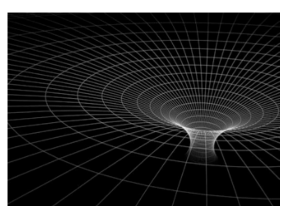
**neutron star**



**black hole by fallback**



**black hole**



**Failed SN (unnovae)**

?

**Long GRB + Type Ib/c SN**

# Looking into the “engine” of a CCSN

Understanding CCSN is one of the primary problems in relativistic astrophysics.

- through observations of neutrinos
  - so far only SN1987A
- through observations of gravitational waves
  - still to be achieved
  - would provide a kind of Rosetta stone.
- through numerical simulations
  - already a 50+ year old effort
  - extremely complex and computationally expensive 6D radiation-hydrodynamics problem
  - ~50 million CPU hours / simulation

# Gravitational waves from CCSN

Einstein quadrupole formula

(adequate for CCSN with PNS formation; Shibata+ 2005, Reisswig+ 2011)

$$h_{ij} = \frac{2G}{Rc^4} \frac{\partial^2 Q_{ij}}{\partial t^2} \sim \frac{R_S}{R} \frac{v^2}{c^2}$$

$$R_S = 3\,\mathrm{km},\ v/c = 0.1,\ R = 10\,\mathrm{kpc} \ \rightarrow \boxed{h \sim 10^{-20}}$$

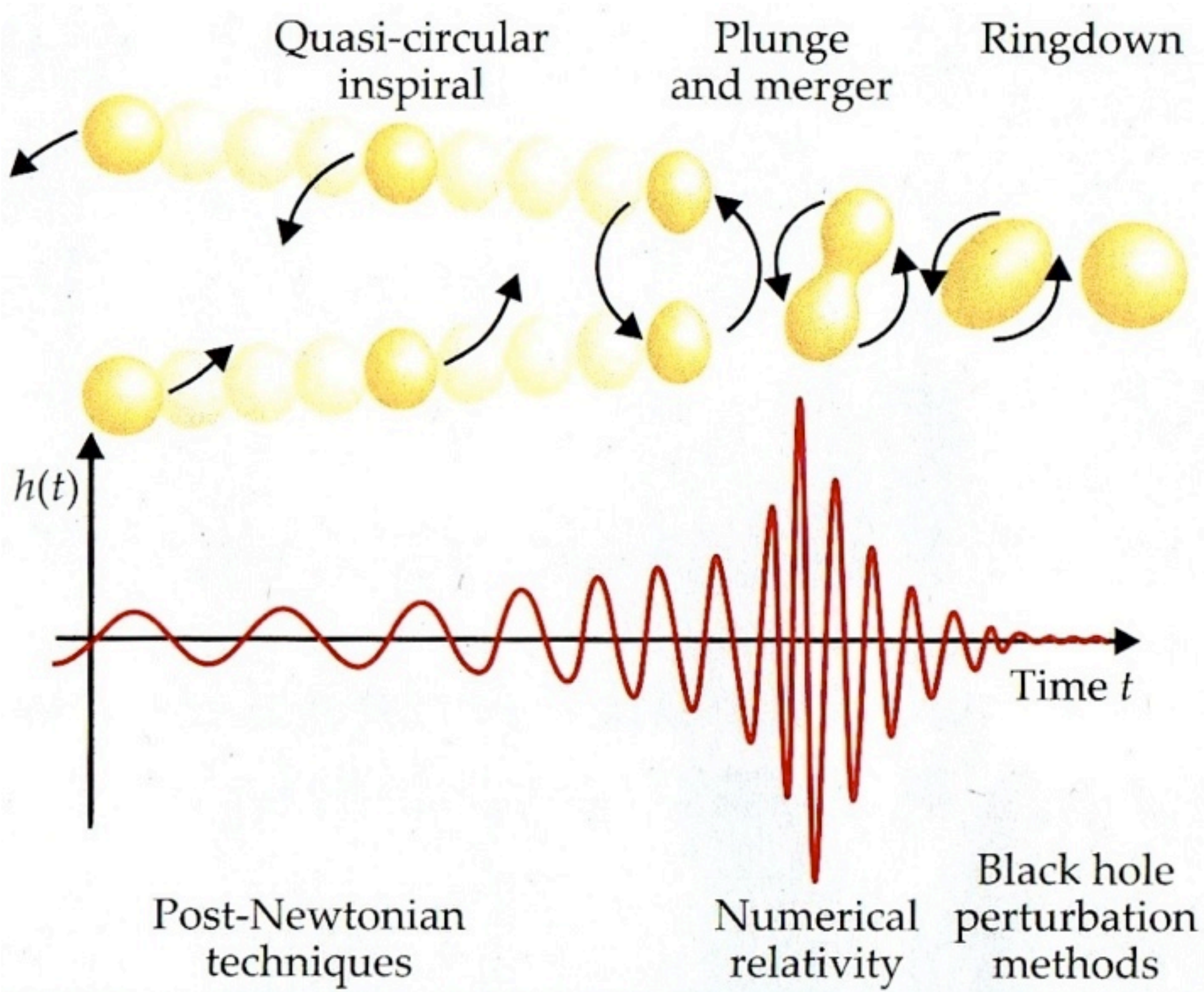
time-dependent mass-energy quadrupole moment in CCSN due to

- convection in proto-neutron star
- convection in neutrino heated hot bubble
- anisotropic neutrino emission
- any other non-radial instability (e.g. SASI)

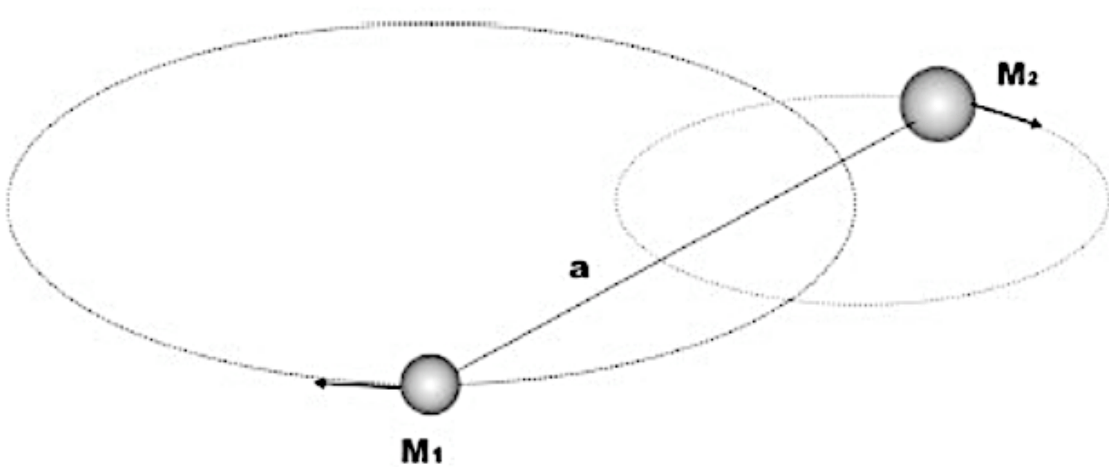
**generically produced by any CCSN**

and due to rotation and magnetic fields

# Mergers of compact binaries



# Orbital dynamics of compact binaries




Let us consider two bodies of masses $M_1$ and $M_2$ separated distances $a_1$ and $a_2$, respectively, from the CM. Let us also consider circular orbits in the xy plane. We can define:

$$a = a_1 + a_2$$

orbital separation

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

reduced mass

$$M = M_1 + M_2$$

total mass

If the objects separation is large, the speeds involved will be small and the orbit can be described with Newtonian dynamics. The orbital frequency can be computed using Kepler's third law:

$$\Omega = \sqrt{\frac{GM}{a^3}}$$

The quadrupole of the system can be computed from the equations of the orbit. The only non-zero components are

$$Q_{xx} = -Q_{yy} = \frac{1}{2}\mu a^2 \cos(2\Omega t)$$
$$Q_{xy} = \quad Q_{yx} = \frac{1}{2}\mu a^2 \sin(2\Omega t)$$

Since the speeds are small, we can use the quadrupole formula to obtain the amplitude of the wave

$$h_{xx} = -h_{yy} = -\frac{1}{r}\frac{G}{c^4}\mu a^2\Omega^2 \cos(2\Omega t)$$
$$h_{xy} = \quad h_{yx} = -\frac{1}{r}\frac{G}{c^4}\mu a^2\Omega^2 \sin(2\Omega t)$$

Note that the wave frequency is twice the orbital frequency. The luminosity is

$$L_{\mathrm{GW}} = \frac{32}{5}\frac{G}{c^5}\mu a^4\Omega^6 = \frac{32}{5}\frac{G^4}{c^5}\frac{M^3\mu^2}{a^5}$$

And the total energy of the system is

$$E_{\mathrm{total}} = \frac{1}{2}\Omega^2(M_1 a_1^2 + M_2 a_2^2) - \frac{GM_1M_2}{a} = -\frac{1}{2}\frac{G\mu M}{a}$$

In GR, the binary system loses energy due to the emission of gravitational waves. Therefore, the distance between both objects is increasingly smaller and the frequency increasingly larger.

$$L_{\rm GW} = -\frac{dE_{\rm total}}{dt} = \frac{1}{2}\frac{G\mu M}{a^2}\frac{da}{dt} \longrightarrow \dot{a} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^2}{a^3}$$

The variation of the orbital frequency will increase according to

$$\frac{\dot{\Omega}}{\Omega} = -\frac{3}{2}\frac{\dot{a}}{a} = \frac{96}{5}\frac{G^3}{c^5}\frac{\mu M^2}{a^4}$$

We can define the “chirp mass” $M_c = \mu^{3/5}M^{2/5}$ and express the variation of the orbital frequency as

$$\dot{\Omega} = \frac{96}{5}\Omega^2\left(\frac{GM_c\Omega}{c^3}\right)^{5/3}$$

The separation between the two objects can be integrated, which results in

$$a(t) = a_0\left(1-\frac{t}{\tau_0}\right)^{1/4} \quad \text{with} \quad \tau_0 := \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^2}$$

If the two objects are separated by a distance $a_0$ , they will merge after a time $\tau_0$

# Example: GW150914

$$\left.\begin{array}{l} M_1 = 29 M_\odot \\ M_2 = 36 M_\odot \end{array}\right\} \Rightarrow M = 65 M_\odot \,,\; \mu = 16 M_\odot \,,\; M_c \sim 28 M_\odot$$

Schwarzschild radii of the two black holes

$$R_{S,1} = 86\,\mathrm{km},\; R_{S,2} = 106\,\mathrm{km}$$

| $a$ | $\omega$ [Hz] | $T = 2\pi/\Omega$ | $\tau_0$ | $L_{GW}$ [erg/s] | $L_{GW}$ [$M_\odot c^2$/ ciclo] | |
|---|---|---|---|---|---|---|
| 1 AU | $1.6 \times 10^{-6}$ | 45 days | $10^{14}$ yr | $10^{28}$ | $3 \times 10^{-20}$ | (1) |
| $10^5$ km | $9.5 \times 10^{-2}$ | 67 s | 22 yr | $10^{44}$ | $4 \times 10^{-9}$ | |
| $10^4$ km | 2.9 | 2.13 s | 0.81 h | $10^{49}$ | $1.4 \times 10^{-5}$ | |
| $10^3$ km | 93 | 67 ms | 0.29 s | $10^{54}$ | 0.04 | (2) |
| 550 km | 228 | 27 ms | 27 ms | $10^{55}$ | 0.36 | (3) |

(1) $\tau_0 >$ age of the Universe

(2) highest sensitivity of LIGO

(3) last orbit $(T = \tau_0)$

$\omega_{\max} \approx 250\,\mathrm{Hz}$ (GW150914)

rough estimate yields correct order of magnitude

# Black hole perturbations

For black holes, the quadrupole formula is not valid, as $M/R \sim 1$

The common approach to study the gravitational-wave emission from BHs is to use perturbation theory.

Let us consider a Schwarzschild BH

$$ds^2 = g^{\mathrm{BH}}_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

and linear perturbations of this metric

$$g_{\mu\nu} = g^{\mathrm{BH}}_{\mu\nu} + h_{\mu\nu} \quad \text{such that} \quad \frac{|h_{\mu\nu}|}{|g^{\mathrm{BH}}_{\mu\nu}|} \ll 1$$

In this case, the Ricci tensor and the Einstein equations read:

$$R_{\mu\nu} = R^{\mathrm{BH}}_{\mu\nu} + \delta R_{\mu\nu} = \delta R_{\mu\nu}$$

$$-\delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu} = 0\,, \quad \delta\Gamma^{\beta}_{\mu\nu;\beta} = \frac{1}{2} g^{\mathrm{BH}\alpha\beta}(h_{\mu\alpha,\nu} + h_{\nu\alpha,\mu} - h_{\mu\nu,\alpha})$$

The solution for the perturbations admits an expansion in tensor harmonics which, in the TT gauge reads

$$h^{\mathrm{TT}}_{\mu\nu} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [a^{lm}(r,t) T^{\mathrm{E2},lm}_{\mu\nu}(\theta,\varphi) + b^{lm}(r,t) T^{\mathrm{B2},lm}_{\mu\nu}(\theta,\varphi)]$$

The previous solution is **not** a multipolar expansion and is valid close to the BH. In addition, far from the source the matching with the multipolar expansion is straightforward, which permits to compute the gravitational waves once the equations are solved in the near zone.

Since we deal with **linear** perturbations, each term in the expansion corresponds to a vibrational mode of the BH. Modes are decoupled due to the linearization.

- The **E2** part of the solution corresponds to **polar perturbations** and are described by the **Zerilli equations**.
- The **B2** part of the solution corresponds to **axial perturbations** and are described by the **Regge-Wheeler equations**.

In both cases, a single scalar describes the dynamics of the perturbations:

$$\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial r_*^2} + \tilde{V}(r) Z = 0 \qquad \text{Zerilli eq.}$$

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + V(r) Q = 0 \qquad \text{Regge-Wheeler eq.}$$

$$r_* \equiv r + 2M \ln(r/2M - 1)$$

$$V(r), \tilde{V}(r) \propto \left(1 - \frac{2M}{r}\right) f(r)$$

effective potentials; depend on BH properties.

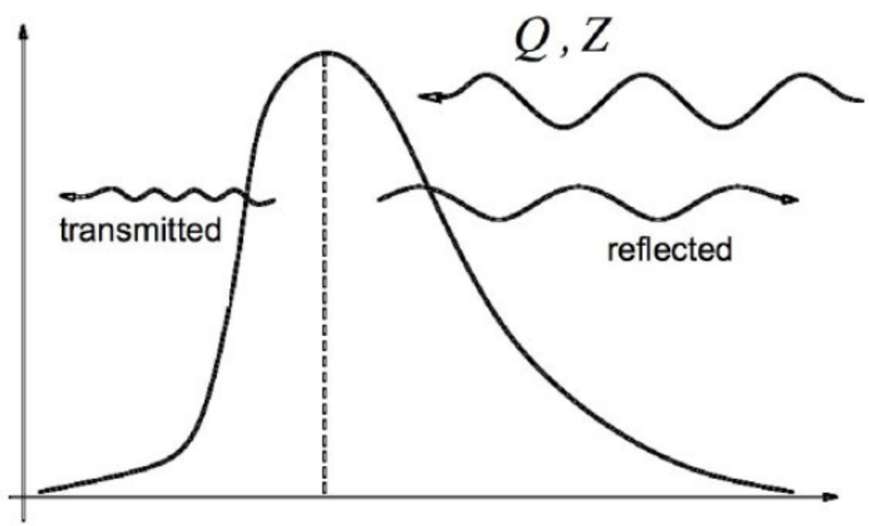
Similar formalism for **rotating (Kerr) BH**: **Teukolsky's master equation.**

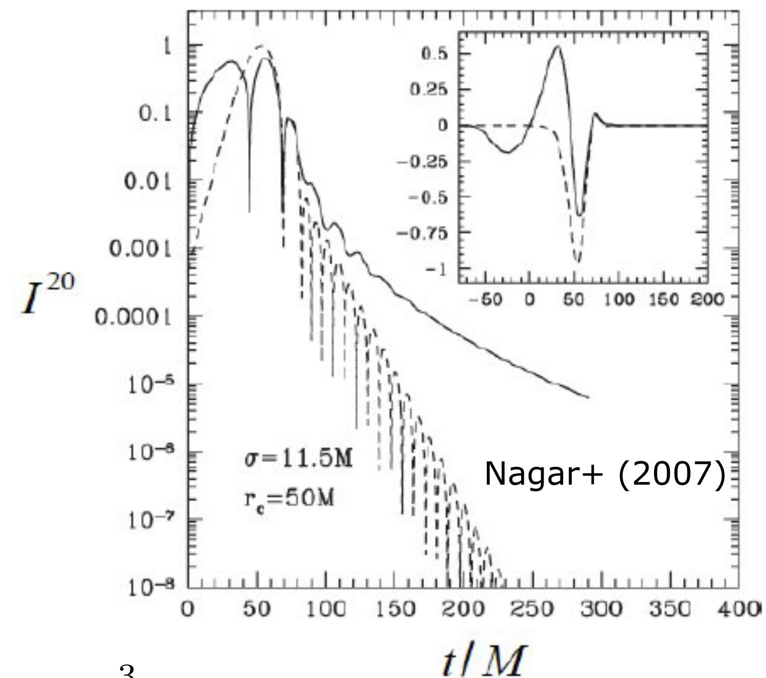
All these equations are similar to the Klein-Gordon equation with a potential. In general, they must be solved numerically.

The solutions have the following **two main properties**:

• Due to the effective potential part of the perturbation is **transmitted** (absorbed by the BH) and part is **reflected** (emission of gravitational waves). A perturbed BH does not absorb fully the energy of the perturbation but always reemits part as GWs (sheds "hair").

• Each mode of oscillation (**quasi-normal mode**) emits gravitational waves with a characteristic frequency. By measuring this frequency, BH properties can be inferred (mass and spin).

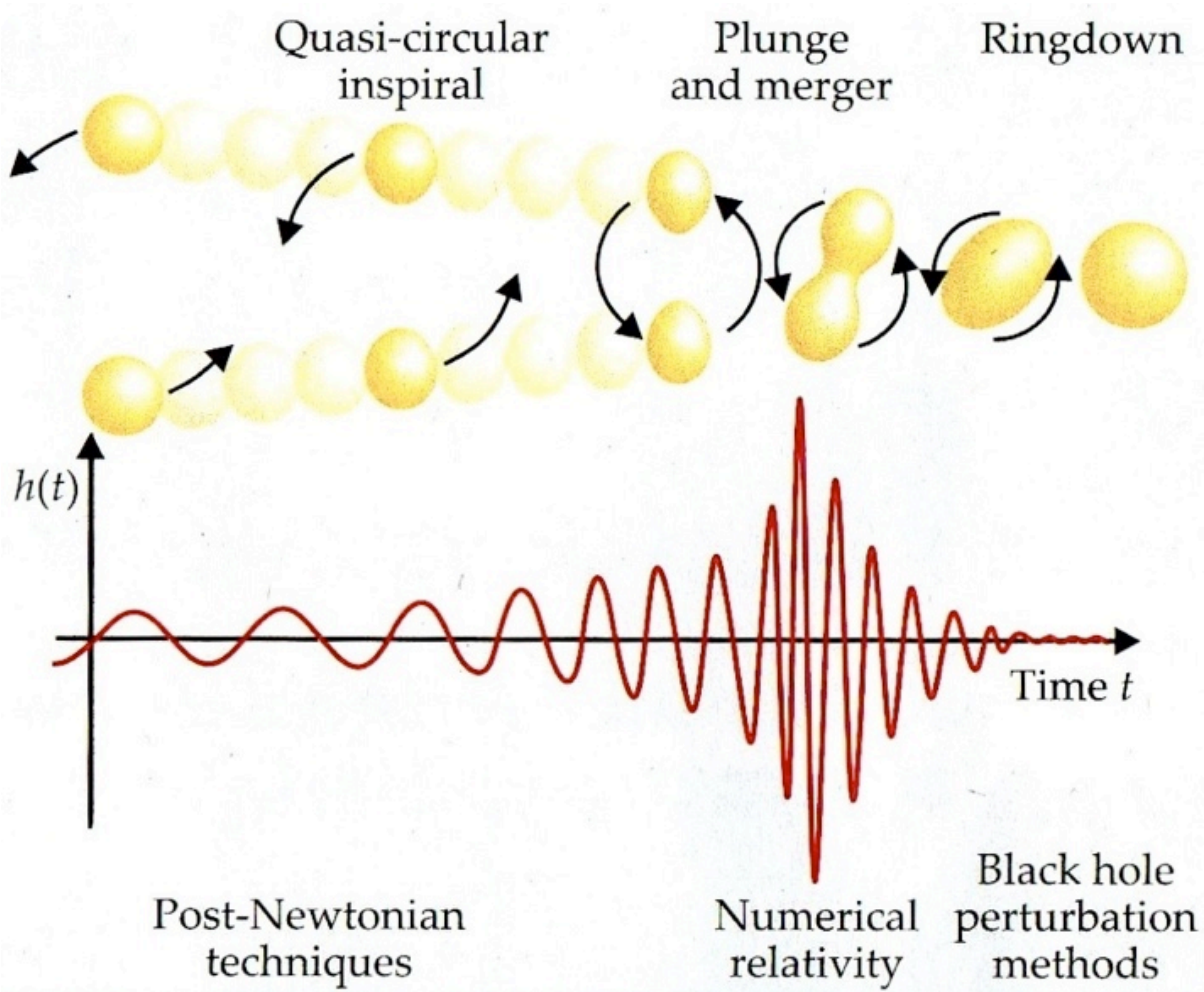



Example: $(l, n) = (2, 0)$ QNM Schwarzshild BH has oscillation frequency $\sigma \approx (0.37 - 0.089i)\frac{c^3}{MG}$

GW150914: $3M_{sun}$ emitted in the merger. The final $62M_{sun}$ BH will oscillate with a frequency of ~191 Hz that will decay exponentially in a ring-down time ~21 ms.

# How to *deal* with the merger?



# Einstein's equations and Numerical Relativity

The dynamics of the gravitational field is described by Einstein's field equations:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

These equations relate the **spacetime geometry** (left-hand side) with the **distribution of matter and energy** (right-hand side): *"Matter tells spacetime how to curve, and spacetime tells matter how to move."*

Einstein's equations are a system of 10 nonlinear, coupled, partial differential equations in 4 dimensions.

When written with respect to a general coordinate system they may contain hundreds of terms ...

Plenty of **exact solutions** of Einstein's equations, but very **few have astrophysical significance.** Exact solutions have only been found when adopting simplifying symmetries:

- **Schwarzschild solution** (static and spherically symmetric)
- **Kerr solution** (stationary and axisymmetric)
- **Cosmological solution** (isotropic, homogeneous, or both)

When studying more complex systems with astrophysical significance (gravitational collapse, mergers of compact binaries) unfeasible to solve Einstein's equations in an exact way.

**Numerical Relativity** began in the mid 1960s from the need to study such kind of problems, aiming at trying to solve the field equations with supercomputers using numerical approximations.

Numerical Relativity's main **goal**: provide templates of the gravitational radiation produced in astrophysical sources to assist detection in LIGO and Virgo.

# Procedure to derive the 3+1 equations

## Cauchy Problem (IVP)

$$\gamma_{ij} \qquad K_{ij}$$

1. Foliation of the 4-dim spacetime with 3-dim spatial hypersurfaces defined by a scalar function, the temporal coordinate. This geometrical construction defines a unit normal vector to the hypersurfaces.

2. Split of 4-dim spacetime tensors into their temporal and spatial parts, using the normal vector and the spatial metric.

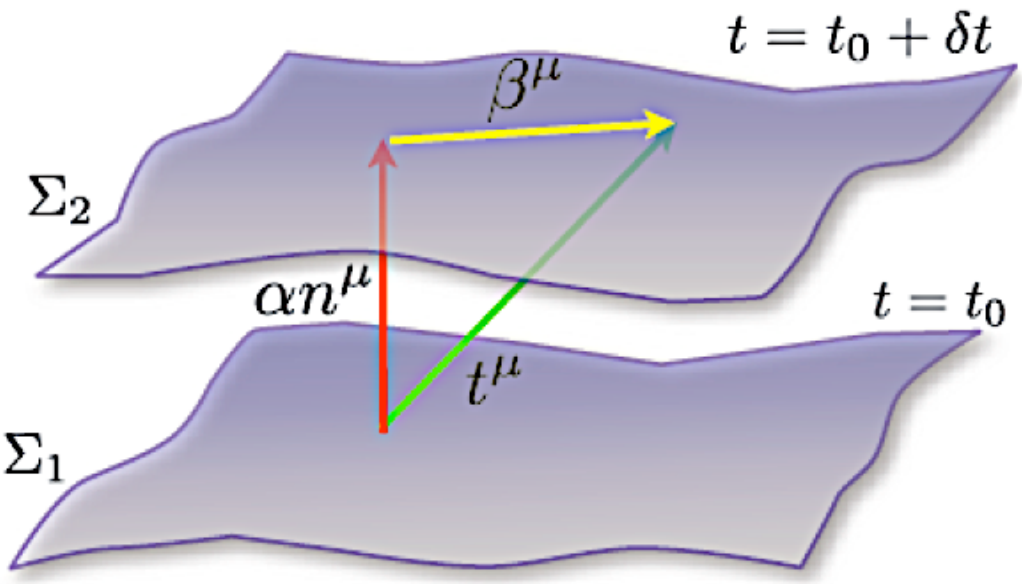
3. Re-writing of Einstein's equations using such split tensors.

4. Choice of a natural direction for the time evolution.

5. Choice of a coordinate basis to express all equations.

# Geometrical aspects of the 3+1 line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$




Therefore:

the lapse function measures the proper time between two adjacent hipersurfaces

$$d\tau^2 = -\alpha^2(t, x^j)dt^2$$

the shift vector relates the spatial coordinates between two adjacent hypersurfaces

$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)dt$$

the spatial metric measures distances between points on each hypersurface

$$dl^2 = \gamma_{ij}dx^i dx^j$$

# Einstein's equations in 3+1 form

Using the projection operator and the normal vector, Einstein's equations can be separated in **three groups:**

- Normal projection (1 equation; energy or Hamiltonian constraint)

$$n^\alpha n^\beta (G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

- Mixed projections (3 equations; momentum constraints)

$$P[n^\alpha (G_{\alpha\beta} - 8\pi T_{\alpha\beta})] = 0$$

- Projection onto the hypersurface (6 equations; evolution of the extrinsic curvature)

$$P(G_{\alpha\beta} - 8\pi T_{\alpha\beta}) = 0$$

## Intrinsic and extrinsic curvature of spatial hypersurfaces:

Intrinsic curvature given by the 3-dimensional Riemann tensor defined in terms of the 3-metric $\gamma_{ij}$ .

Extrinsic curvature $K_{ij}$ measures the change of the vector normal to the hypersurface as it is parallel-transported from one point in the hypersurface to another.

Projection operator: $P^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta} + n^{\alpha} n_{\beta}$

Unit normal vector: $n^{\mu} = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), \;\; n_{\mu} = (-\alpha, 0), \;\; n^{\mu} n_{\mu} = -1$

$$K_{\alpha\beta} \equiv -P^{\mu}_{\alpha} P^{\nu}_{\beta} \nabla_{\mu} n_{\nu} = -(\nabla_{\alpha} n_{\beta} + n_{\alpha} n^{\mu} \nabla_{\mu} n_{\beta}))$$

Substituting the form of the normal vector in the definition of the extrinsic curvature, we get:

$$K_{ij} = \frac{1}{2\alpha}(-\partial_t \gamma_{ij} + \nabla_i \beta_j + \nabla_j \beta_i)$$

# 3+1 Formulation (Cauchy)

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

## (ADM) Evolution equations:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \quad (*) \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha \left( R_{ij} + K\ K_{ij} - 2K_{im}K^m_j \right) + \beta^m \nabla_m K_{ij} \\
&\quad + K_{im}\nabla_j \beta^m + K_{mj}\nabla_i \beta^m - 8\pi\alpha \left( T_{ij} - \frac{1}{2}\gamma_{ij}T^m_m + \frac{1}{2}\rho\gamma_{ij} \right)
\end{aligned}
$$

## Constraint equations:

$$
\begin{aligned}
R + K^2 - K^{ij}K_{ij} &= 16\pi\rho \\
\nabla_i \left( K^{ij} - \gamma^{ij}K \right) &= 8\pi S^j
\end{aligned}
$$

$$(*)\ (\partial_t - \mathcal{L}_\beta)\gamma_{ij} = -2\alpha K_{ij}$$

## Cauchy problem (IVP):

- Specify $\gamma_{ij}$ , $K_{ij}$ at t=0 subjected to the constraint equations.
- Specify coordinates through $\alpha$, $\beta^i$
- Evolve the data using EE and the definition of $K_{ij}$

# Definitions:

$\nabla_i$ Covariant derivative with respect to induced 3-metric

Ricci tensor $$R_{ij} = \partial_n \Gamma^n_{ij} - \partial_j \Gamma^n_{in} + \Gamma^n_{mn}\Gamma^m_{ij} - \Gamma^n_{jm}\Gamma^m_{in}$$

Christoffel symbols $$\Gamma^i_{jk} = \frac{1}{2}\gamma^{in}\left(\frac{\partial \gamma_{nj}}{\partial x^k} + \frac{\partial \gamma_{nk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^n}\right)$$

Scalar curvature $$R = R_{ij}\gamma^{ij}$$

Trace of extrinsic curvature $$K = K_{ij}\gamma^{ij}$$

Matter fields $$\begin{cases} \rho & \equiv & T^{\mu\nu} n_\mu n_\nu = \rho h W^2 - P \\ S^i & \equiv & -\perp^i_\mu T^{\mu\nu} n_\nu = \rho h W^2 v^i \\ S_{ij} & \equiv & \perp^\mu_i \perp^\nu_j T^{\mu\nu} = \rho h W^2 v_i v_j + \gamma_{ij} P \\ S & \equiv & \rho h W^2 v_i v^i + 3P \end{cases}$$

# ADM vs Maxwell

The ADM equations look rather cryptic and complicated. The analogy between these equations and Maxwell's equations helps to better understand the ADM system.

In electromagnetism, the relevant quantities are the electric and magnetic fields, the charge density, and the charge current density, that is: $\mathbf{E}, \mathbf{B}, \rho_e, \mathbf{J}$

Maxwell's equations also split into evolution equations

[Ampere] $\partial_t \mathbf{E} = \nabla \times \mathbf{B} - 4\pi \mathbf{J} \quad \Leftrightarrow \quad \partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i$

[Faraday] $\partial_t \mathbf{B} = -\nabla \times \mathbf{E} \quad \Leftrightarrow \quad \partial_t B_i = -\epsilon_{ijk} D^j E^k$

and constraint equations [Gauss]

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e \quad \Leftrightarrow \quad \partial_i E^i = 4\pi \rho_e$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Leftrightarrow \quad \partial_i B^i = 0$$

For Maxwell's equations it is also possible to prove that if the constraint equations are satisfied at the initial time, then the evolution equations preserve that property.

To highlight even more the similarities, let us introduce the vector potential

$$A_\mu = (\Phi, A_i) \qquad \text{such that} \qquad B_i = \epsilon_{ijk} D^j A^k$$

Therefore, the evolution part of Maxwell's equations reads:

$$\partial_t A_i = -E_i - D_i \Phi$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

to compare with the evolution equations of the ADM system

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$\begin{aligned} \partial_t K_{ij} = & - \quad D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K^{kj} + K K_{ij}) \\ & - \quad 8\pi\alpha (R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_\beta K_{ij}) \end{aligned}$$

Nowadays, ADM equations are hardly used.

While the ADM equations have no peculiarities from a mathematical point of view, numerical experience has shown that they are not suitable for a numerical approach.

In particular, it has been shown that the ADM system is weakly hyperbolic and, therefore, constitutes an ill-posed Cauchy (IVP) problem.

In practice, the ADM system is prone to the appearance of numerical instabilities that destroy the solution (exponentially unstable growing modes).

However, the stability properties of numerical implementations can be improved by introducing new auxiliary functions and rewriting the ADM equations in terms of those new functions.

# Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation

$$\mathcal{D}_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}$$

$$\mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t\phi = -\frac{1}{6}\alpha K$$

$$\mathcal{D}_t\tilde{A}_{ij} = e^{-4\phi}[-\nabla_i\nabla_j\alpha + \alpha(R_{ij} - S_{ij})]^{\mathrm{TF}} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j)$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i\nabla_j\alpha + \left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho + S)\right]$$

$$\mathcal{D}_t\tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right)$$
$$-\partial_j\left(\beta^l\partial_l\tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l\beta^l\right)$$

These equations are known as BSSN equations or simply the conformal, traceless formulation of Einstein's equations.

Kojima, Nakamura & Oohara (1987); Shibata & Nakamura (1995); Baumgarte & Shapiro (1999)

Although not evident, the BSSN equations constitute a strongly hyperbolic system and have a structure that resembles that of a first-order in time, second-order in space system:

$$\Box\phi = 0 \;\Leftrightarrow\; \begin{cases} \partial_t \phi = \psi \\ \partial_t \psi = \partial^i \partial_i \phi \end{cases}$$

scalar wave equation

$$\begin{cases} \partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij} \\ \\ \partial_t \tilde{A}_{ij} \propto D^i D_i \tilde{\gamma}_{ij} \end{cases}$$

conformal traceless formulation

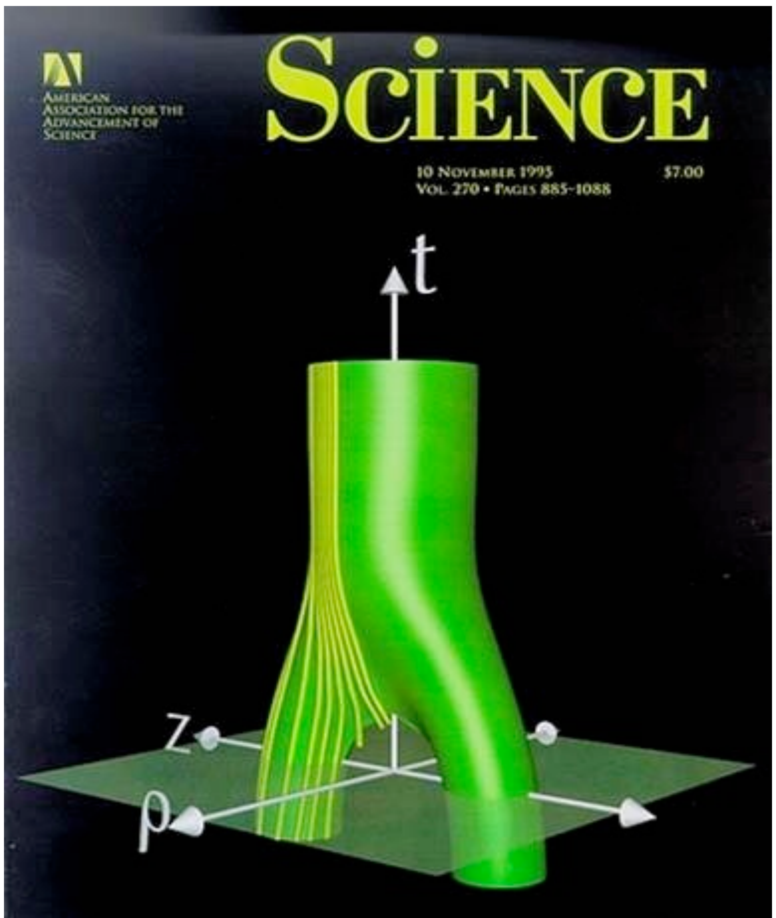
BSSN is nowadays the standard 3+1 formulation in Numerical Relativity.

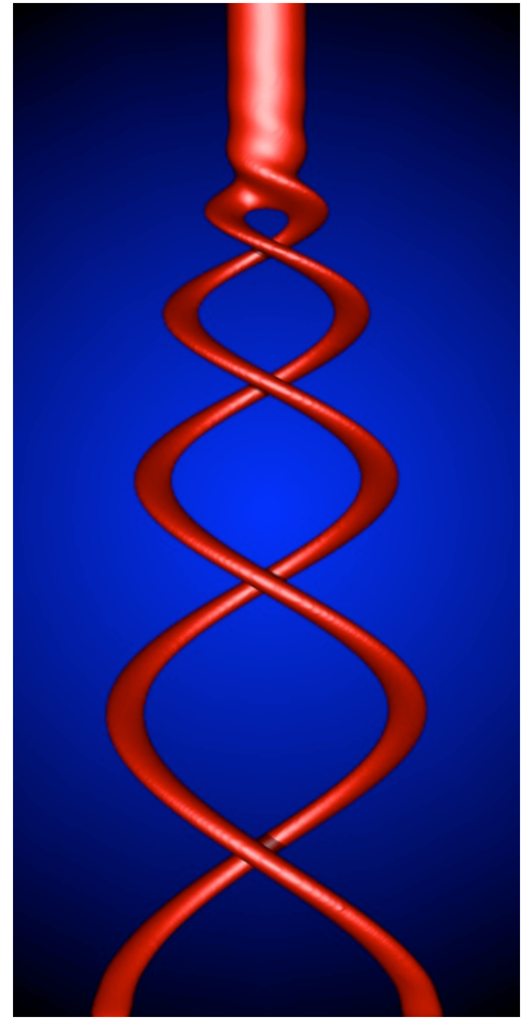
Long-term stable numerical simulations have been possible for strongly gravitating systems as neutron stars (isolated and in binary systems) and black holes (isolated and in binary systems).

# BBH simulations: State of the art

1995: Pair of pants
(Head-on collision)

BBH Grand Challenge Alliance

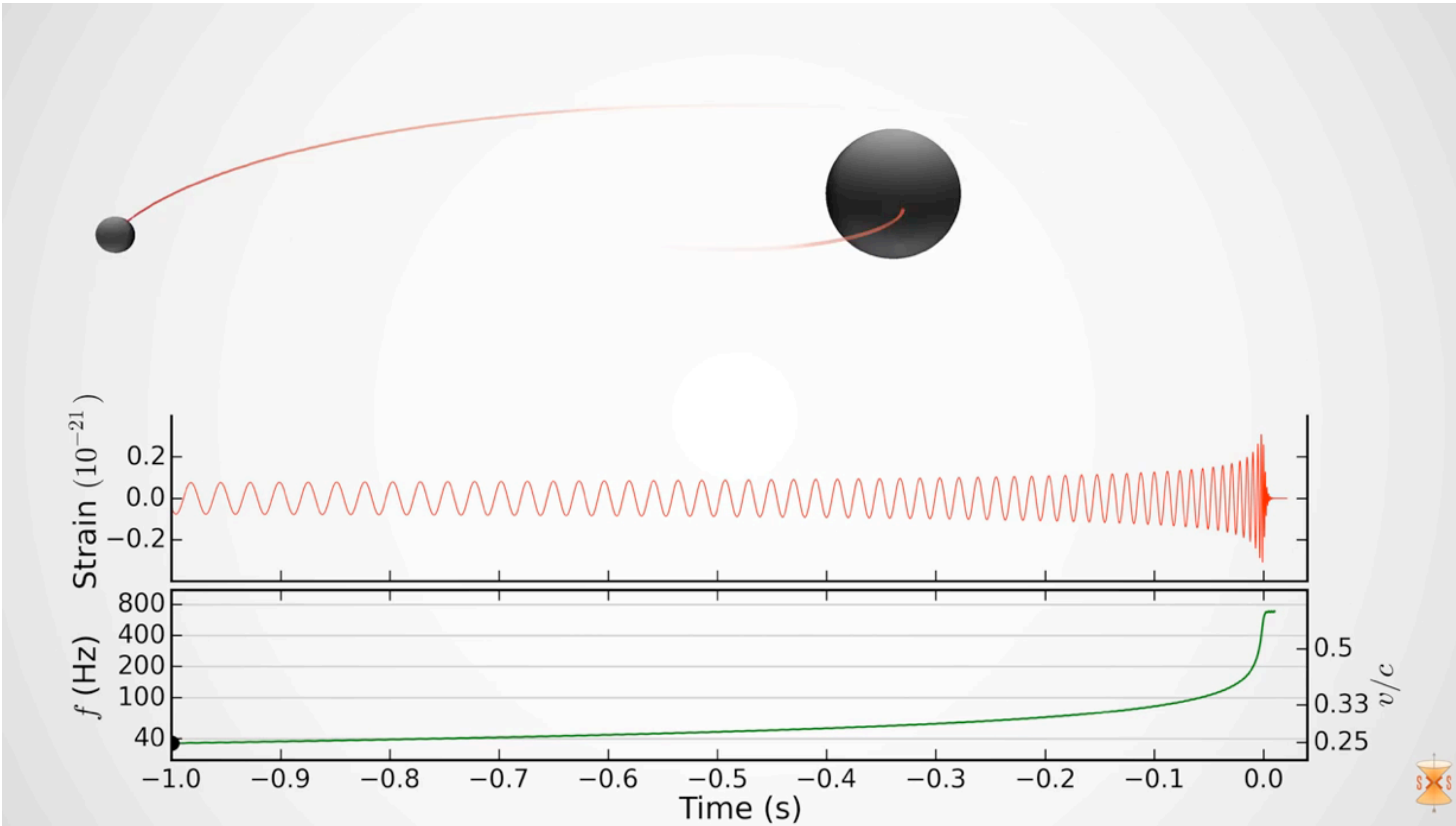


FSU-Jena Numerical Relativity Group

2007: Pair of twisted pants
(spiral & merge)

# Numerical Relativity
## Binary Black Hole Merger

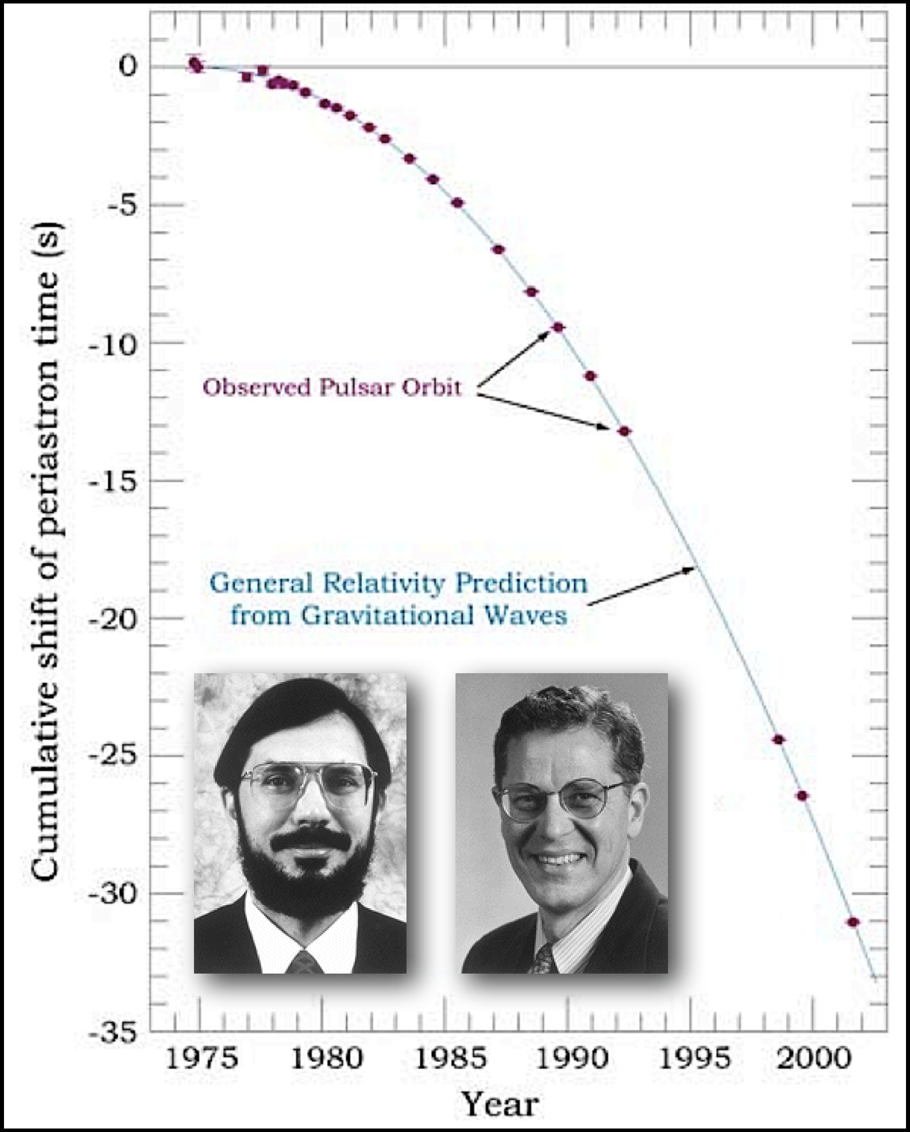



(SXS)

Binary neutron stars **observed** in the Galaxy.

Black hole binaries? **Hypothetical** system.

# PSR B1913+16: Hulse-Taylor binary pulsar




(J. Bell 1967)

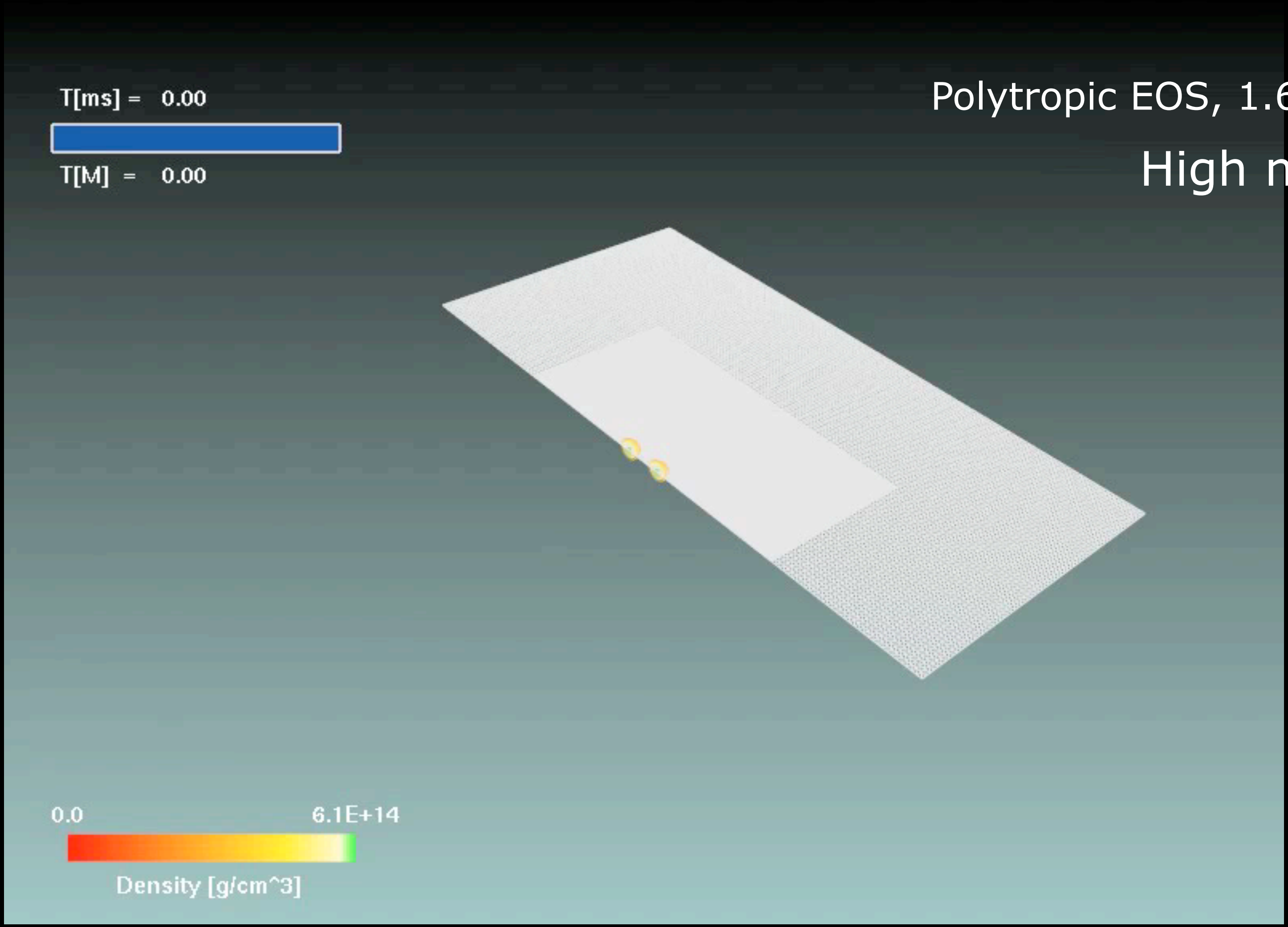
What is a **pulsar**?

- very compact star
- rapidly-rotating star
- strong magnetic field

The orbital evoultion of PSR B1913+16 is that of a system that emits gravitational radiation according to General Relativity.

(Nobel Prize 1993)

# Equal-mass BNS merger




Polytropic EOS, 1.6 $M_{sun}$

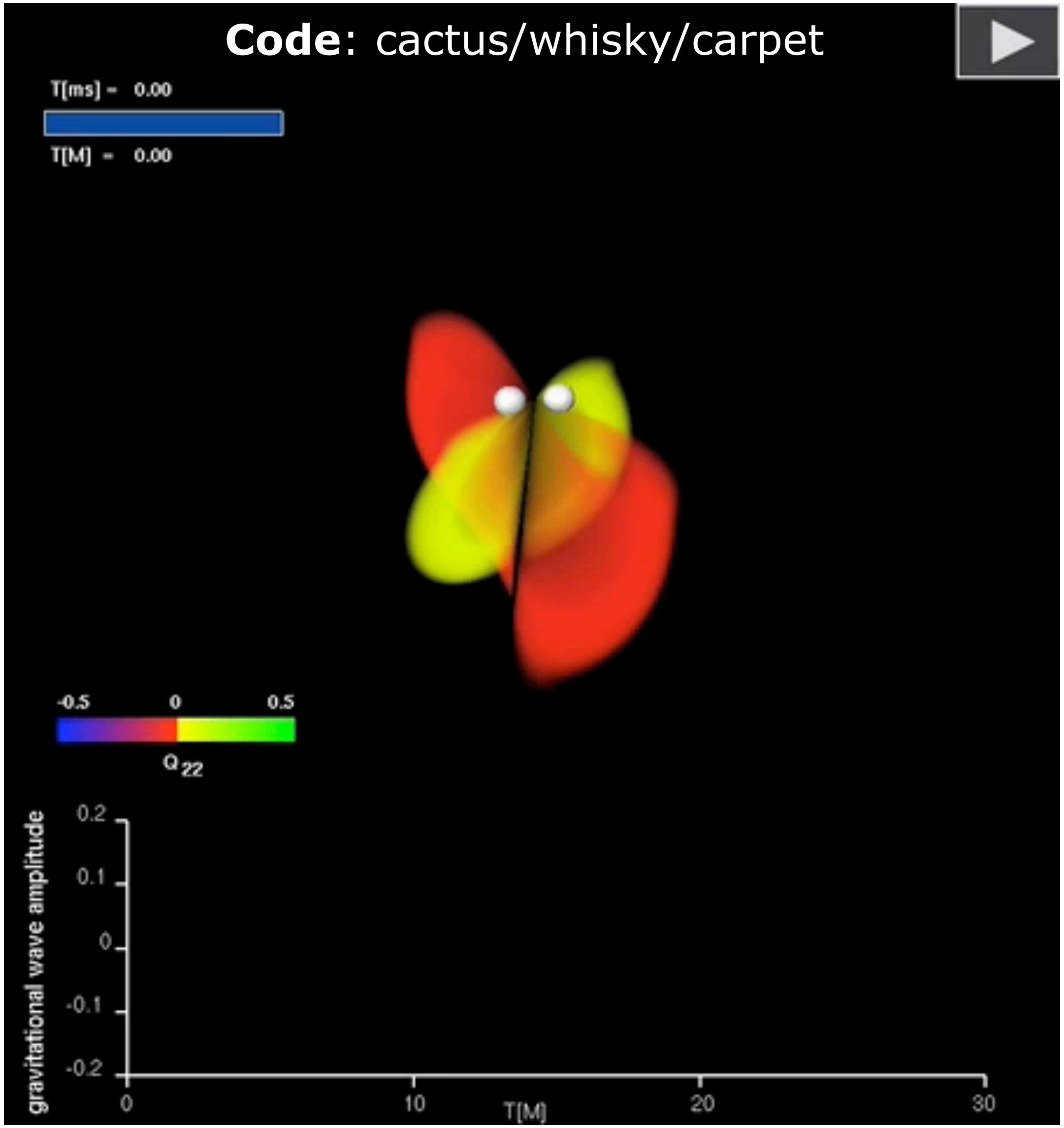
High mass

A hot, low-density torus is produced orbiting around the BH. This is what is expected in short GRBs.

(Baiotti et al 2008)

# Gravitational radiation from the merger

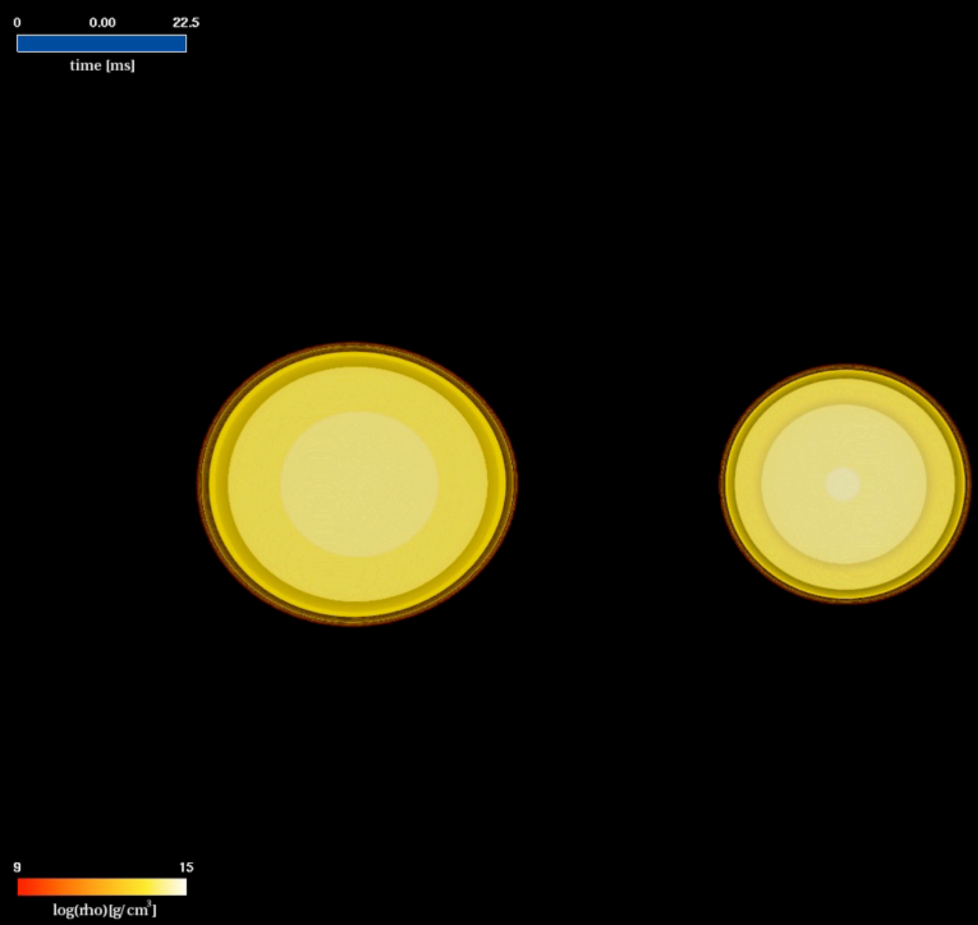
Dynamics




Signal amplitude

(AEI)

# Unequal-mass BNS merger



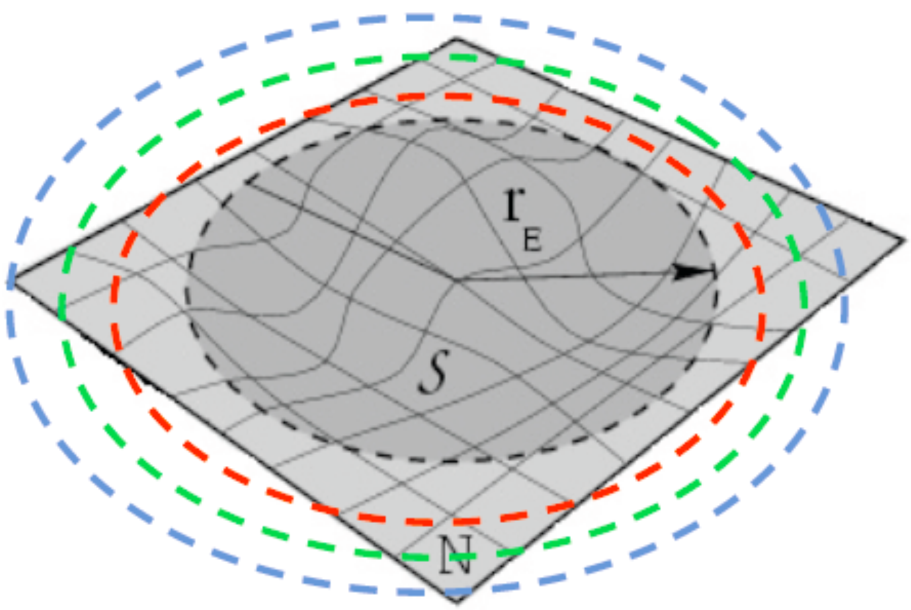

A significantly more massive torus is formed in this case.

(Rezzolla et al 2010)

# Gravitational wave extraction in (3+1) NR

**First approach:** perturbations on a Schwarzschild background expanding spatial metric into a tensor basis of Regge-Wheeler harmonics. Allows for extracting gauge-invariant wavefunctions given spherical surfaces of constant coordinate radius.

**Second approach:** projection of the Weyl tensor onto components of a null tetrad. At a sufficiently large distance from the source and in a Newman-Penrose tetrad frame, the gravitational waves in the two polarizations can be written in terms of the Weyl scalar.




In both approaches observers are located at various positions from the source (nested spheres), where Weyl scalars are computed or where the metric is decomposed in tensor spherical harmonics to compute gauge invariant perturbations of a Schwarzschild black hole.

# Gravitational wave extraction in (3+1) NR

**First approach:** perturbations on a Schwarzschild background expanding spatial metric into a tensor basis of Regge-Wheeler harmonics. Allows for extracting gauge-invariant wavefunctions given spherical surfaces of constant coordinate radius.

$$
\begin{aligned}
Q^{\times}_{lm} &= \sqrt{\frac{2(l+2)!}{(l-2)!}}\left[c_1^{\times lm} + \frac{1}{2}\left(\partial_r - \frac{2}{r}c_2^{\times lm}\right)\right]\frac{S}{r} \\
Q^{+}_{lm} &= \frac{1}{\Lambda}\sqrt{\frac{2(l-1)(l+2)}{l(l+1)}}(l(l+1)S(r^2\partial_r G^{+lm} - 2h_1^{+lm}) + 2rS(H_2^{+lm} - r\partial_r K^{+lm}) + \Lambda r K^{+lm})
\end{aligned}
$$

These functions satisfy the Regge-Wheeler $(Q^{\times}_{lm})$ and Zerilli $(Q^{+}_{lm})$ wave eqs.

$$
\begin{aligned}
(\partial_t^2 - \partial_{r*}^2)Q^{\times}_{lm} &= -S\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]Q^{\times}_{lm} \\
(\partial_t^2 - \partial_{r*}^2)Q^{+}_{lm} &= -S\left[\frac{1}{\Lambda^2}\left(\frac{72M^3}{r^5} - \frac{12M(l-1)(l+2)}{r^3}\left(1 - \frac{3M}{r}\right)\right) + \frac{l(l^2-1)(l+2)}{r^2\Lambda}\right]Q^{+}_{lm}
\end{aligned}
$$

with the definitions: $S = 1 - \dfrac{2M}{r}$ $\quad \Lambda = (l-1)(l+2) + \dfrac{6M}{r}$

$$r^* = r + 2M\ln(r/2M - 1)$$

# Gravitational wave extraction in (3+1) NR

Far from the source and assuming the spacetime resembles that of a Schwarzschild BH, the GWs in the two polarizations can be written in terms of the odd and even-parity gauge invariant perturbations of a Schwarzschild BH as

$$h = h_+ - ih_\times = \frac{1}{\sqrt{2}r} \sum_{l,m} \left( Q^+_{lm} - i \int_{-\infty}^{t} Q^\times_{lm}(t'\, dt' \right) Y^{-2}_{lm} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

**Second approach:** projection of the Weyl tensor onto components of a null tetrad.

At a sufficiently large distance from the source and in a Newman-Penrose tetrad frame, the GWs in the two polarizations can be written in terms of the Weyl scalar $\Psi_4$ as

$$h = h_+ - ih_\times = - \int_{-\infty}^{t} dt' \int_{-\infty}^{-t'} \Psi_4 \, dt''$$

The Weyl scalar $\Psi_4$ can be computed explicitly in terms of projections of the 4-Riemann tensor onto a null Newman-Penrose tetrad.

Further reading: Newman & Penrose, *An Approach to Gravitational Radiation by a Method of Spin Coefficients*, J. Math. Phys. **3**, 566 (1962); http://dx.doi.org/10.1063/1.1724257

# Estimates of the GW amplitude

Amplitude of gravitational radiation produced by a source at a distance *r*:

$$h_{ij} \sim \frac{G}{c^4 r} \frac{d^2 Q_{ij}}{dt^2} \sim 7 \times 10^{-50} \frac{E}{r}$$

The amplitude of a gravitational wave is expressed in terms of a dimensionless quantity *h* that measures the relative displacement in the length *L* between two test masses:

$$h = \frac{\Delta L}{L}$$

# Example

Gravitational wave source at the Virgo cluster radiating the energy equivalent to one solar mass per unit time.

$$r = 20\,\mathrm{Mpc} \sim 6 \times 10^{25}\,\mathrm{cm}$$
$$E = M_{\odot} c^2 \sim 2 \times 10^{54}\,\mathrm{erg}$$
$$\longrightarrow \boxed{h \sim 10^{-21}}$$

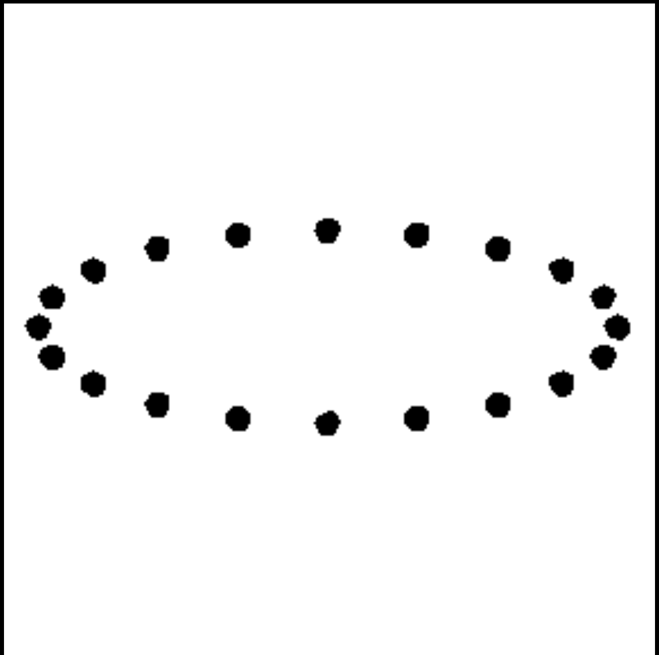
Therefore, a detector with a baseline of 10 km could measure changes in the relative length

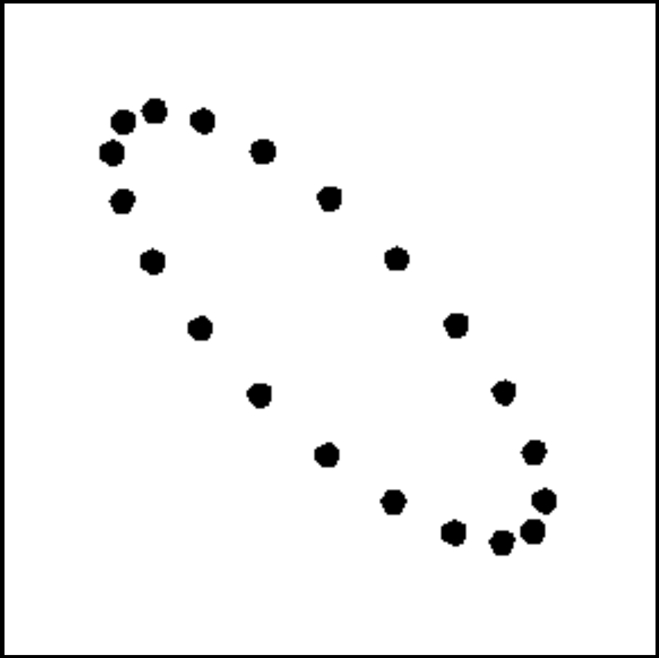
$$\Delta L = h\,L \sim 10^{-15}\,\mathrm{cm} \;= 0.01\,\mathrm{fm}$$

of the order of a hundreth of a Fermi, that is, **of the order of a hundreth of the size of a proton.**

**This estimate is *very* optimist.** The emission of gravitational radiation in astrophysical sources is much smaller.

# Tidal field





Length variations smaller than size of an atomic nucleus

4. 000 000 000 000 000 000 001 km
3. 999 999 999 999 999 999 999 km

# Gravitational luminosity estimate

Astrophysical source of mass $M$ and radius $R$

Quadrupole moment: $Q \sim \varepsilon M R^2$ $\qquad$ $\varepsilon$ source asymmetry

$$\mathcal{L} \sim \epsilon^2 \frac{G}{c^5} \frac{M^2 R^4}{T^6}$$

$T$ characteristic time of evolution of the source

Expressing the mass of the source in terms of its Schwarzschild radius and introducing a characteristic speed

$$R_{\mathrm{S}} = \frac{2GM}{c^2} \qquad v = \frac{R}{T}$$

$$\mathcal{L} \sim \epsilon^2 \frac{G}{c^5} \frac{c^4 R_{\mathrm{S}}^2}{4G^2} \frac{v^6}{R^2} \sim \epsilon^2 \frac{c^5}{G} \left( \frac{R}{R_{\mathrm{S}}} \right)^{-2} \left( \frac{v}{c} \right)^6$$

Prime astrophysical sources of gravitational radiation: **compact objects** with matter at **relativistic speeds**.

$$\mathcal{L} \sim \varepsilon^2 \, \frac{c^5}{G} \left(\frac{R}{R_S}\right)^{-2} \left(\frac{v}{c}\right)^6$$

$$\mathcal{L} \sim 10^{59} \, \mathrm{erg\, s}^{-1} \sim 10^{23} \mathcal{L}_\odot \sim 10^{52} \, \mathrm{W}$$

**as the light emitted by $10^{50}$ 100 W light bulbs!**

The highest frequencies are obtained for stellar mass compact objects such as neutron stars or black holes.

Gravitational radiation

$$f \leq 10^4 \, \mathrm{Hz}$$

within the audible range

$$[20 \, \mathrm{Hz}, 20 \, \mathrm{kHz}]$$

Electromagnetic radiation

$$f \geq 10^7 \, \mathrm{Hz}$$

[visible light

$f \sim 10^{11} \, \mathrm{Hz}$]