A gentle introduction to neutrino physics

Pasquale D. Serpico
LAPTh, UMR 5108, 9 chemin de Bellevue - BP 110, 74941 Annecy-Le-Vieux, France

I. FIRST LECTURE

A. Intro

In one class of radioactive decays (β decays) one observes a nuclear transmutation of $\Delta A = 0$, $\Delta Z \neq 0$. These decays involve the emission or absorption of (at least) one electron, or the emission of (at least) one positron. A representative case is Tritium decay

$$^3\mathrm{H} \to {}^3\mathrm{He} + e \quad (observed)\,. \tag{1}$$

If it is all what happens, there are *serious problems with the above-mentioned observations!* For example

- Conservation of energy and momentum
  It appears to be violated, since the observed spectrum of the e is *continuous*, of type shown in Fig. 1. For example, by neglecting the $^3\mathrm{He}$ recoil one should expect $E_e \simeq m_{3\mathrm{H}} - m_{3\mathrm{He}} = m_e + 18.6\,\mathrm{keV}$, while e of different energies are measured.
- Angular momentum conservation (and related "statistics" problem)
  Spin of $^3\mathrm{H}$ is $1/2$, just like the spin of $^3\mathrm{He}$ and e. But according to quantum mechanics $1/2 \otimes 1/2 = 0 \oplus 1$.

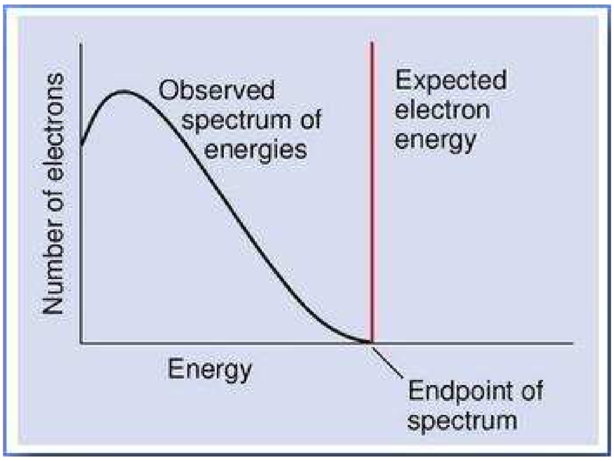

FIG. 1: A typical observed $\beta-$decay spectrum (black), compared with expectations from e.g. Eq. (1) (red).

Pauli proposed the following *desperate remedy*:

- a new (unobserved) particle should exist, the "neutron" (later dubbed *neutrino* ="little neutral one", by Fermi)
- it should have no electric charge (and very small magnetic moment as well).
- it should have spin 1/2 (fermion).
- should be (almost) massless (for sure much smaller than the proton, but turned out also than the electron).

Based on this, provided that the underlying dynamics is universal (same for all nuclei) the basic element of the β transitions in nuclei is thus $n \to p + e + \bar{\nu}$ (modern notation).

But how can one hope to test Pauli's idea? We need a(t least a rough) theory, allowing to compute neutrinos interaction rates.

B. Fermi theory

In 1934, Fermi built a "theory" out of the Pauli intuition (rejected by the journal *Nature* as too speculative). The transition rate for β decay given by "Fermi Golden Rule"

$$T_{i\to f} = \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f) \tag{2}$$

where $\rho =$ is the density of final states around available final state energy E_f, and the matrix element between initial and final state is

$$H_{fi} = \langle p\, e\, \nu | H | n \rangle = G \int (\phi_\nu^* \phi_e^* \psi_p^*) \psi_n \mathrm{d}^3\mathbf{x}\,, \tag{3}$$

which assumes a "contact interaction" of strength G (*Fermi constant*) among the particles involved, of dimensions Energy $\times$ Volume. The hypothesis of contact interaction implies that the electron and the neutrino (which do not participate to strong nuclear interactions) can be considered to have free particle wavefunctions (in an arbitrary normalization volume V):

$$\phi_e \simeq \frac{1}{\sqrt{V}} e^{-\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}}\,, \tag{4}$$

$$\phi_\nu = \frac{1}{\sqrt{V}} e^{-\frac{i\mathbf{q}\cdot\mathbf{x}}{\hbar}} \tag{5}$$

Evidently some correction is expected at least for the electron, which has e.m. interaction. In the above approx., we obtain

$$H_{fi} \simeq \frac{G}{V} \int \psi_p^* \psi_n e^{\frac{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}{\hbar}} \mathrm{d}^3\mathbf{x}\,. \tag{6}$$

The typical energy scale of nuclear decays is of the order of MeV (or even smaller), hence momenta of neutrinos and electrons are at most of few MeV$/c$. The wavefunctions are non-vanishing only over nuclear sizes of the order of fm$\simeq 10^{-15}$ m. Once taking into account that $\hbar\, c \sim \mathcal{O}(200)$ MeV fm, we conclude that the argument of the exponential is very small and an expansion of the following type makes sense:

$$H_{fi} = \underbrace{\frac{G}{V} \int \psi_p^* \psi_n d^3\mathbf{x}}_{allowed\, transition} + \underbrace{i\,\frac{G}{V} \int \psi_p^* \psi_n \frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}{\hbar} \mathrm{d}^3\mathbf{x}}_{first\, forbidden\, transition} + \ldots \tag{7}$$

For the simplest case,

$$H_{fi} = \frac{G}{V} M_{fi}\,, \quad \text{where } M_{fi} \equiv \int \psi_p^* \psi_n d^3\mathbf{x}\,. \tag{8}$$

One expects $M_{fi} \sim \mathcal{O}(1)$ (apart for electromagnetic corrections) for the free neutron decay or if the final state nucleon simply occupies the initial nucleon state (e.g. in nuclear shell model), but will be in general depend on complicated nuclear physics.

Concerning ρ, it is the result of integration over all the momentum variables but for the final energy E_f, which is the sum of the electron kinetic energy, T_e, the total invisible energy of the neutrino E_ν, and the energy of the recoiling nucleus/nucleon. We consider the ($\simeq$free) electron (p) and neutrino (q) momenta as independent variables (the recoiling nucleon/us momentum being fixed by momentum conservation, and being actually negligible), so that

$$\frac{d\rho}{dp}(E_f) = \frac{d\Pi_e}{dp} \frac{d\Pi_\nu}{dE_f} = \frac{V\, 4\pi\, p^2}{h^3} \frac{V\, 4\pi\, q^2}{h^3} \frac{dq}{dE_f} \tag{9}$$

We now express "neutrino" variables in terms of the electron (visible) ones. In particular, we need $q^2\, dq/dE_f$; we exploit the relation

$$q^2\, c^2 + m_\nu^2 c^4 = (E_f - T_e)^2 \Longrightarrow c^2 q dq = (E_f - T_e) dE_f \tag{10}$$

to obtain (note that it is fully general, also for massive neutrinos, and would follow even from considering $m_\nu = 0$ in the above!)

$$q^2 \frac{dq}{dE_f} = \frac{q(E_f - T_e)}{c^2}\,. \tag{11}$$

In the limit of massless neutrino (and neglecting recoil of daughter nucleus, so that one can also estimate $E_f \simeq Q$)

$$q = \frac{E_f - T_e}{c}\,, \tag{12}$$

and

$$\frac{d\rho}{dp} = \frac{V^2\,16\pi^2\,p^2(E_f - T_e)^2}{h^6\,c^3}\,, \tag{13}$$

from which the decay rate can be written as a function of electron momentum as

$$\frac{d\Gamma}{dp} = \frac{G^2|M_{fi}|^2}{2\pi^3\hbar^7 c^3}p^2(E_f - T_e(p))^2\,, \tag{14}$$

or kinetic energy (remember $\sqrt{p^2c^2 + m_e^2c^4} = m_e\,c^2 + T_e$) as

$$\frac{d\Gamma}{dT_e} = \frac{G^2|M_{fi}|^2}{2\pi^3\hbar^7 c^3}p(T_e)^2(E_f - T_e)^2\frac{dp}{dT_e} = \frac{G^2|M_{fi}|^2}{2\pi^3\hbar^7 c^3}p(T_e)(m_e^2c^4 + T_e)(E_f - T_e)^2\,. \tag{15}$$

The final state interaction effect between e (charge q in absolute electron charge units) and the daughter nucleon/nucleus of charge Z is accounted for by multiplying times a *Fermi function* $F(q\,Z, T_e)$ (which reduces to the "Sommerfeld factor" you may be more familiar with in non-relativistic limit). The further correction due to non-leading terms in the series of Eq. (7) is eventually accounted for via a shape factor function $S(p,q)$ (which related the nuclear element to light final state momenta).

1. *Determination of G*

By integrating the above expressions over the final state electron momentum (or Energy) one obtains the decay width:

$$\Gamma = \frac{G^2|M_{fi}|^2}{2\pi^3\hbar^7 c^3}\int_0^Q p^2(Q - T_e(p))^2 F(q\,Z,p)dp = \frac{G^2|M_{fi}|^2\,m_e^5c^7}{2\pi^3\hbar^7}\underbrace{f(q\,Z,Q)}_{Fermi\ integral}\,, \tag{16}$$

with the latter function tabulated. A subset of β decays with short lifetimes (large widths) is characterized by nuclear transitions $0^+ \to 0^+$ for which the matrix element can be simply estimated (*superallowed*). They can be used to determine $\underline{G = 8.962 \times 10^{-5}\,\mathrm{MeV\,fm}^3}$. The fact that a common value of G fits all these simple decays also suggest universality of the mechanism underlying the Fermi theory (at least within nuclear physics uncertainties).

C. Estimate of neutrino cross-sections

Let's use the above knowledge to estimate the cross-section for the process

$$\bar{\nu} + p \to n + e^+ \tag{17}$$

at typical MeV energies. We get

$$\sigma = \frac{\text{prob. reaction per proton per unit time}}{\text{neutrino flux}} = \frac{\Gamma}{\phi}\,. \tag{18}$$

The flux corresponding to a neutrino number density $1/V$ is $\phi = c/V$. The final energy E_f is comparable to the produced positron energy if one can neglect the nucleon recoil energies, and if one can also neglect the $p - n$ mass difference and the m_e, it is also almost comparable to the incoming antineutrino energy.

The probability at the numerator writes

$$\Gamma = \frac{2\pi}{\hbar}\frac{G^2}{V^2}|M_{fi}|^2\rho(E_f) = \frac{2\pi}{\hbar}\frac{G^2}{V^2}|M_{fi}|^2\frac{V\,4\,\pi\,p^2dp}{h^3dE_f} = \frac{2\pi}{\hbar}G^2|M_{fi}|^2\frac{4\,\pi\,p\,E}{h^3c^2\,V}\frac{dE}{dE_f} \tag{19}$$

hence, assuming $E \sim E_f$, one obtains the following estimate

$$\sigma = \frac{1}{\pi\hbar^4\,c^4}G^2|M_{fi}|^2\,c\,p(E)\,E \sim \left(\frac{E}{10\,\mathrm{MeV}}\right)^2 10^{-42}\,\mathrm{cm}^{-2} \tag{20}$$

Despite this extremely small cross section, neutrinos could eventually be detected, see [1–3] [1]. Neutrinos are now "routinely" detected, both from natural and artificial sources (see Exp. Neutrino lectures). Actually, the fact that neutrinos are so weakly interacting proves them interesting astrophysical messengers as well: they can travel unimpeded from dense environments teaching us about core-collapse supernovae, for example, or energetic extragalactic events (as PeV=1000 TeV scale) whereas even photons are absorbed (and charged cosmic rays do not retail directionality being charged!). All this, provided one can build detectors big and sensitive enough to detect some astrophysical neutrinos, of course!

D. Beyond Fermi Theory: a few facts and notions

1. Lepton number. . . and more generations ("who ordered that?" I. Rabi)

If an e^- is emitted in a weak decay, then an *anti*-neutrino is emitted together with it. The "anti" label means that if/when the particles emitted in such a decay are detected with reactions on nuclei, they will produce e^+ but are never observed to produce e^-.

More formally, if we attribute a quantum number to the electron, which we call "lepton number" L and normalize it arbitrarily by setting $L(e^-) = 1$, its antiparticle (positron) will have $L(e^+) = -L(e^-) = -1$, as well as $L(e^-) = L(\nu_e)$ and $L(\bar{\nu}_e) = L(e^+)$ and we summarize the above observation by saying that in all reactions *lepton number is conserved* (the algebraic sum of all L's is conserved in any process, where we implicitly assumed that baryons, photons, etc. have $L = 0$).

In the '30 to '50, many particles were discovered among byproducts of collisions of energetic nuclei from outer space (*cosmic rays*) hitting Earth atmosphere. Among these particles, one of mass intermediate between electrons and protons (*mesotrons*) was looked for as the "mediator" of strong nuclear interactions predicted by Yukawa (later dubbed pions, π).

Surprisingly, it was found that the most abundant of these "mesotrons" were not interacting strongly, but just electromagnetically with nuclei, like "heavy electrons" [8]. These particles (muons μ) were the first hint of some "multiple family" structure in the particle zoo, and the main mechanism of their production is from the decay of the "Yukawa particle", the pion, via $\pi^- \to \mu^- + \bar{\nu}$ (or the antiparticle version $\pi^+ \to \mu^+ + \nu$).

In 1962, it was discovered that the kind of neutrinos associated to muons (such as those emitted in association with them at pion decay) were of a different kind, in the sense that in following interactions they could produce muons, not electrons [9]. So, the "lepton number" associated to muons (and related muon neutrinos ν_μ as well as their antiparticles) has a different nature and one introduced a new quantum number L_μ, which appeared to be *separately conserved* (i.e. in addition to L_e now reserved to the *first generation* of leptons) in particle interactions.

Later on, the story was replicated with another heavier partner, the τ (discovered by Perl et al. in 1975 [10]) and the related ν_τ (directly observed by DONUT collaboration in 2000 [11], albeit their existence had already been indirectly inferred both by theoretical and experimental arguments), with similar properties involving the new quantum number L_τ.

2. Helicity

Helicity is the projection of the spin $\mathbf{S}$ (or total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$) onto the direction of momentum

$$\Sigma = \mathbf{J} \cdot \hat{\mathbf{p}} = \mathbf{S} \cdot \hat{\mathbf{p}} \,, \tag{21}$$

the reason for the equality being that the orbital angular momentum is anyway orthogonal to the momentum. It follows also that the measured helicity of a spin S particle will range from -S to +S (in units of $\hbar$).

For massless particles, helicity is frame-independent, but for massive ones it is not (in fact, it is replaced by a more abstract concept, the one of *chirality*, related to the Poincaré group representation according to which particles transform).

3. Parity

Parity transformation (spatial mirror inversion) is defined as

$$P: \{t, \mathbf{x}\} \to \{t, -\mathbf{x}\} \,, \tag{22}$$

[1] Pauli confessed to the astronomer Baade (quoted by Hoyle in 1967): "I've done a terrible thing today, something which no theoretical physicist should ever do. I have suggested something that can never be verified experimentally." That was one of the rare times Pauli was proven wrong.

and more in general changes real vectors and leaves invariant pseudovectors (like the angular momentum $\mathbf{J}$):

$$P:\ \mathbf{J} \to -\mathbf{x} \times -\mathbf{p} = \mathbf{J}\,. \tag{23}$$

For a wavefunction

$$P\psi(t,\mathbf{x}) \to e^{i\theta}\psi(t,-\mathbf{x}) \tag{24}$$

and it also obeys

$$P^2\psi(t,\mathbf{x}) = \psi(t,\mathbf{x}) \Longrightarrow P^2 = \mathsf{I}\,, \tag{25}$$

as well as the unitarity requirement (to preserve state normalization)

$$P^\dagger P = \mathsf{I} \Longrightarrow P^\dagger = P\,. \tag{26}$$

The latter condition is hermiticity, which ensures that its eigenvalues are observables.

A certain interaction ("theory") described by the hamiltonian H is invariant under P if this operator commutes with its hamiltonian,

$$[H,P] = 0\,. \tag{27}$$

In that case, reactions mediated by H conserve parity, and parity is also a good quantum number for a bound state mediated by this interaction.

4. *Parity violation*

It came as a *shocking discovery* that *parity is violated in weak reactions involving neutrinos*!

Triggered by a 1956 paper by Lee and Yang [4] questioning parity conservation in weak interactions, Wu et al. [5] tested parity conservation in 1957 by studying the transition:

$$\underbrace{{}^{60}\mathrm{Co}(J=5)}_{\Longrightarrow} \to {}^{60}\mathrm{Ni}^*(J=4) + \underbrace{e^-}_{\Rightarrow} + \underbrace{\bar{\nu}_e}_{\Rightarrow}\ . \tag{28}$$

The ^{60}Co spins were aligned by a magnetic field at 0.02 K. Note that the both the electron and antineutrino spin (J $=$ 1/2) have to be in the direction of the ^{60}Co spin because angular momentum is conserved and orbital angular momentum is zero for a point interaction as the one we are dealing with (s-wave). On the other hand, you can easily see that either an electron momentum parallel or antiparallel to its spin is naively allowed (provided that the antineutrino has opposite momentum, so that momentum conservation is ensured) Suprisingly, electrons were not found to be emitted isotropically, rather they were found to be distributed as

$$I(\theta) \propto 1 - \left(\frac{\sigma\cdot\mathbf{p_e}}{E_e}\right)\,, \tag{29}$$

where σ is a unit vector in the direction of the electron spin and θ is the angle of momentum relative to the ^{60}Co spin: electrons are emitted preferentially with direction (momentum) opposite to the parent nucleus spin (hence negative helicity, since they must have spin aligned with nucleus spin). The distribution of Eq. (29) is seen to violate parity, since applying P would leave σ (and E_e) unchanged, but invert $\mathbf{p_e}$.

Analogous experiments with β^+ decays (positron emission) show the preferential positive helicity for the positrons.

This experiment also suggest that the antineutrino must be emitted with positive helicity (and similarly, neutrinos have negative helicity), which were proven respectively in 1960 [6] and 1957 [7].

5. *Theory modification*

As a consequence of the discovery of parity violation, Fermi theory had to be modified. Its *Lorentz structure* was eventually established to be of the $V-A$ type, for those of you who know what that means, an important step in what was to become the standard model of particle physics.

Another important aspect is that the original Fermi theory (Fig. 2) became to be appreciated as the low-energy limit of a theory where massive particle mediators are exchanged between particles (see Fig. 3). The W boson (massive and charged "cousin" of the photon) is interpreted as mediating the interactions we described before. Additionally, other types of reactions (with no change of particle nature, just energy-momentum) are possible, so-called Neutral Current mediated by a "neutral partner" of the W boson, called Z boson (see Fig. 4). Interactions mediated by W are known as Charged Current processes. The effective coupling G we introduced above is known as "Fermi constant", G_F, and related to the microscopic picture as $G_F = 2^{-5/2} g^2/m_W^2$, where g is the coupling in the weak current vertices and m_W the mass of the heavy gauge boson exchanged in Fig. 3.

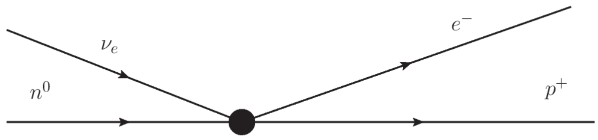

FIG. 2: Graphical sketch of the Fermi contact interaction theory

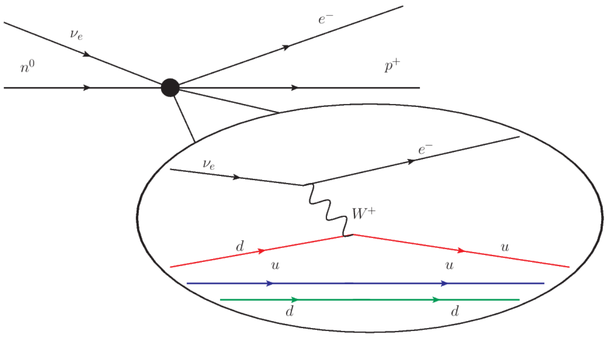

FIG. 3: A "zoom" on the Fermi contact interaction theory revealing the microscopic nature, mediated by a W boson.

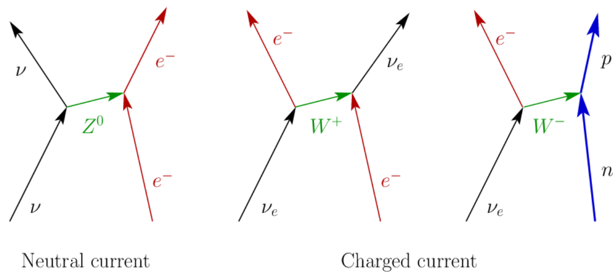

FIG. 4: Diagrams for Charged and Neutral currents to which neutrinos take place.

II. SECOND LECTURE

A. Kinematical Neutrino mass determination (and alternatives, in brief)

If we do not neglect the mass of the neutrino, Eqs. (12) should be replaced by

$$q = \frac{\sqrt{(E_f - T_e)^2 - m_\nu^2 c^4}}{c}, \tag{30}$$

which, if replaced in Eq. (9) together with Eq. (11), leads to the following modified version of Eq. (14) and Eq. (32)

$$\frac{d\Gamma}{dp} = \frac{G^2 |M_{fi}|^2}{2\pi^3 \hbar^7 c^3} p^2 (E_f - T_e(p))^2 \sqrt{1 - \frac{m_\nu^2 c^4}{(E_f - T_e(p))^2}}, \tag{31}$$

$$\frac{d\Gamma}{dT_e} = \frac{G^2|M_{fi}|^2}{2\pi^3\hbar^7c^3} p(T_e)(m_e^2c^4 + T_e)(E_f - T_e)^2 \sqrt{1 - \frac{m_\nu^2 c^4}{(E_f - T_e)^2}} \,. \tag{32}$$

It is evident that the shape of the differential spectrum near the endpoint $E_f \simeq Q$ is qualitatively different if $m_\nu = 0$ or $m_\nu \neq 0$: In general,

$$\left.\frac{d\Gamma}{dT_e}\right|_{T_e=Q} = \begin{cases} \text{finite} & \text{if } m_\nu = 0\,, \\ \infty & \text{if } m_\nu \neq 0\,. \end{cases} \tag{33}$$

This different shape has been crucial in obtaining limit to neutrino masses till now and still is used in the kinematical searches for neutrino mass (KATRIN, MARE). It is customary to plot the (Fermi-)Kurie function

$$K(T_e) \propto \sqrt{\frac{d\Gamma/dT_e}{p(T_e)(m_e^2c^4 + T_e)F(q\,Z, T_e)}}\,, \tag{34}$$

which is expected to be linear in $E_f - T_e$ for vanishing neutrino mass, with a departure from this behaviour for massive neutrinos (see Fig. 5).

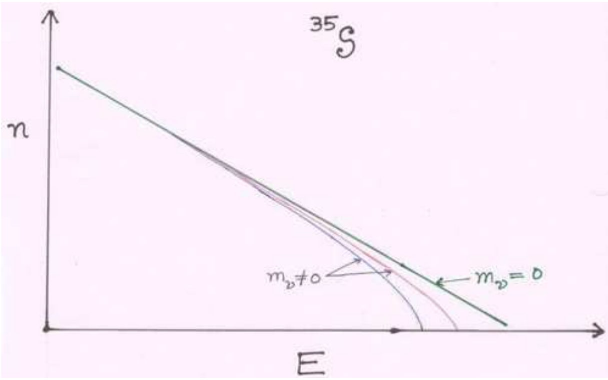

FIG. 5: "Fermi-Kurie" plot: qualitatively different shape of the endpoint of β decay spectrum in presence of non-vanishing neutrino mass.

Let us mention that other ways to obtain information on the neutrino mass are: i) via its *gravitational* effects in cosmology, current bounds are better than $\sum_i m_i < 1\,$eV. ii) through nuclear and particle physics processes (i.e. dynamics) mediated by neutrino mass operators, as the case of neutrinoless double beta decay $(A, Z) \to (A, Z+2) + 2\,e^-$ (but depends on theory assumptions).

B. Evidence for neutrino mass from oscillations

Early observations seemed to have revealed the existence of different neutrino states, which are labelled by their *flavour* $\alpha = e, \mu, \tau$, depending on their *interaction properties* (e.g. the particle they have been produced with). If neutrinos were massive, however, one may conceive to label different states via their masses (i.e. *propagation eigenstates*). Each set of properties (being of one flavour among the possible ones, or having one mass among the possible ones) is exhaustive, so they form two possible bases. In a simple Quantum Mechanics framework, any neutrino state can be written as a superposition of different flavour states, or different mass states [2].

The fact that a propagating state (for now think of it as eigenstate of the "free" hamiltonian) has a well-defined flavour or not is an empirical question. Put more formally, there is no conceptual (consistency) reason why the lepton numbers must be conserved. The probability that a given mass eigenstate interacts by creating an e-flavoured lepton need not to be

[2] Discussion here does not make justice of many conceptual subtleties. For example, in Quantum Field Theory external particles (asymptotic states) are required to be on-shell, which applies only to mass eigenstates. Flavour states are just particular superposition of mass eigenstates associated to internal lines (propagators) attached to weak interaction vertices. Also, from now on we stick to "particle physics" practice in which natural units $\hbar = c = 1$ are used.

1 or 0. Mathematically, we can describe this by writing that the link between the two (if we limit ourselves to 2 flavours) bases eigenvectors as a simple rotation, namely at some arbitrary time $t=0$ (or equivalently space location) we have

$$
\begin{aligned}
|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle && (35)\\
|\nu_\mu\rangle &= -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle && (36)
\end{aligned}
$$

or more compactly as

$$
\nu_\alpha = \sum_j U_{\alpha\, j}\,\nu_j \Longleftrightarrow \nu_k = \sum_\beta U^*_{\beta\, k}\,\nu_\beta \tag{37}
$$

with U a unitary matrix (orthogonal, in the simple example reported above; note that in the second case the proper matrix is the conjugate of the transpose, hence the inverted indices). In the rest of this section we follow the simpler explicit expression, leaving the generalization of Eq. (37) to following subsec. II.B.1.

By definition, $\nu_{1,2}$ are the propagating eigenstates so that, if their masses are $m_{1,2}$, in their rest-frame their evolution in proper time τ_i is given by (henceforth we use natural units with $\hbar = 1\,, c=1$)

$$
\nu_i(\tau_i) = e^{-i m_i\,\tau_i}\nu_j(0)\,. \tag{38}
$$

In the laboratory frame, once can conceptually define the travelled distance L and elapsed time (from source to detection) t related to the above phase by

$$
m_i\,\tau_i = E_i t - p_i L\,, \tag{39}
$$

in terms of the neutrino energy E_i and momentum p_i. For the time being, let us symbolically write the phase above as $\phi_i = \phi_i(t,L)$. Inverting Eqs. (35,36) one has

$$
\begin{aligned}
|\nu_1\rangle &= \cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle\,, && (40)\\
|\nu_2\rangle &= \sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle\,, && (41)
\end{aligned}
$$

from which one sees that a state which is ν_e initially (Eqs. (35)) evolves as

$$
|\nu_e(t,L)\rangle = (\cos^2\theta e^{-i\phi_1} + \sin^2\theta e^{-i\phi_2})|\nu_e\rangle + \sin\theta\cos\theta(e^{-i\phi_2} - e^{-i\phi_1})|\nu_\mu\rangle\,. \tag{42}
$$

It is evident from the above expression that a $\mu-$flavoured component can appear, provided that some *phase-difference* develops between the two states. The issue is how to write it in a physically meaningful way. Let us pause a bit and consider what a realistic experiment measures (having in mind "stationary" measurements, of characteristic timescales much longer that the intrinsic ones). The time t is almost never measured, and some sort of "average" is performed, over timescales much longer than $1/E$ in natural units. Interference effects between different propagating states make phases of the kind $\exp[i(E_1-E_2)t]$ appear. Needless to say, averaging over such kind of phases leads to a vanishing result, *unless* $E_1 = E_2 = E$. Hence, in order to compute an interference phenomenon for a stationary situation (and isolate the neutrino oscillation phenomenology we are interested in), we can limit ourselves to consider only those mass eigenstate components of a beam that have the same energy, E, and can thus contribute coherently [3].

The corresponding momentum p_i can be expressed in the relativistic approximation (always true for detected neutrinos!) as

$$
p_i = \sqrt{E^2 - m_i^2} \approx E\left(1 - \frac{m_i^2}{2\,E^2}\right)\,. \tag{43}
$$

so that

$$
\phi_i(t,L) = E(t-L) + \frac{m_i^2}{2\,E}L\,, \tag{44}
$$

where the phase $E(t-L)$ is common to all states and thus irrelevant.

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[3] This is a heuristic view developed more carefully e.g. in [12, 13], but it does not make justice of long and conceptually interesting subtle points developed at length over decades, which however do not affect the final results derived here: it is enough to say that nobody doubts about the correctness of the final formulae "for all practical purposes". For further discussion see e.g. [14].

One can finally recast Eq. (42) into the form

$$|\nu_e(L)\rangle = (\cos^2\theta e^{-i\frac{m_1^2}{2E}L} + \sin^2\theta e^{-i\frac{m_2^2}{2E}L})|\nu_e\rangle + \sin\theta\cos\theta(e^{-i\frac{m_2^2}{2E}L} - e^{-i\frac{m_1^2}{2E}L})|\nu_\mu\rangle\,, \tag{45}$$

which implies that the probability for measuring it into a muon state at a distance L is the square of the amplitude coefficient, namely

$$P_{e\to\mu}(L) = |\langle\nu_\mu|\nu_e(L)\rangle|^2 = \sin^2(2\theta)\sin^2\left(\frac{(m_2^2-m_1^2)L}{4\,E}\right) = \sin^2(2\theta)\sin^2\left(\frac{\pi\,L}{\ell_{osc}}\right)\,, \tag{46}$$

where we used

$$e^{ia} - e^{ib} = 2\,e^{i\frac{a+b}{2}}\,\frac{e^{i\frac{a-b}{2}} - e^{-i\frac{a-b}{2}}}{2} = 2\,e^{i\frac{a+b}{2}}\sin\left(\frac{a-b}{2}\right)\,, \tag{47}$$

and the second equality introduces the conventional notation of the oscillation length ℓ_{osc}

$$\ell_{osc} = \frac{4\pi\,E\,\hbar}{\Delta m^2\,c^3} = 2.48\left(\frac{E}{\text{GeV}}\right)\left(\frac{\text{eV}^2}{\Delta m^2}\right)\text{km}\,. \tag{48}$$

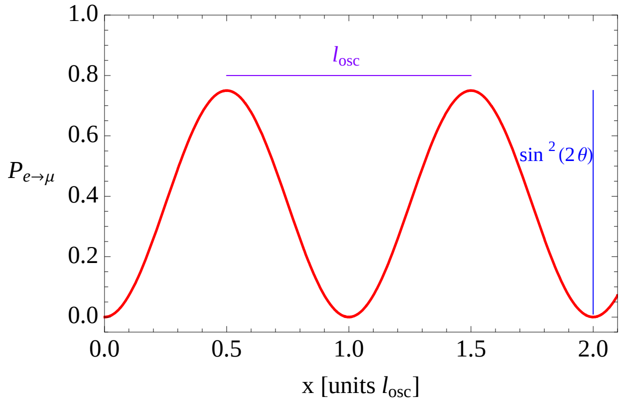

FIG. 6: Typical flavour oscillation pattern, assuming a mixing angle of $\theta = 30°$.

A few comments:

- It is easy to check that $P_{e\to e} = 1 - P_{e\to\mu}$ (unitarity is preserved).
- in the limiting of vanishing mixing $\theta \to 0$, mass and flavour eigenstates coincide and there is no oscillation. Given the weak interaction properties of neutrinos, experiments are usually plagued by low statistics, so it seems that only large mixing angles can be realistically probed.
- the detection of neutrino oscillations would imply that (at least some) neutrinos are massive. (The converse is not true). Yet, the absolute mass scale is not accessible, only neutrino squared mass differences.
- A too poor distance and/or energy resolution would correspond to averaging the $\sin^2(\pi L/\ell_{osc})$ term to $1/2$, in which case neutrino "mixing" may still be detected, but not the typical *oscillatory* pattern.
- For light neutrinos (eV scale or lighter) and energies above the MeV, this quantum oscillation phenomenon happens on macroscopic scales and can be (and has actually been!) tested on "man-made" laboratories. This is how nowadays we know that neutrinos are massive! See Exp. Neutrino lectures.

1. *More general treatment of oscillations*

Following the previous arguments, the evolution of flavour states is more generically written as (apart for an irrelevant, overall phase)

$$\nu_\alpha(L) = \sum_j U_{\alpha\,j}\,e^{-i\frac{m_j^2}{2E}L}\nu_j = \sum_\beta\sum_j U_{\alpha\,j}\,e^{-i\frac{m_j^2}{2E}L}U^*_{\beta\,j}\,\nu_\beta\,, \tag{49}$$

hence the amplitude that a neutrino (initially of flavour α) is in the β state at time t is the coefficient proportional to ν_β in the above, namely

$$\langle\nu_\beta|\nu_\alpha(L)\rangle = \sum_j U_{\alpha\, j}\, e^{-i\frac{m_j^2}{2\,E}L} U^*_{\beta\, j}\,, \tag{50}$$

hence the probability is

$$P_{\alpha\to\beta}(L) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = \left|\sum_j U_{\alpha\, j}\, e^{-i\frac{m_j^2}{2\,E}L} U^*_{\beta\, j}\right|^2 = \sum_{j,k} U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\, e^{-i\frac{m_j^2-m_k^2}{2\,E}L}\,. \tag{51}$$

The double sum can be split in $\sum_{j=k}+\sum_{j>k}+\sum_{k>j}$ and the latter two terms are evidently the sum of two complex conjugate expressions, hence

$$P_{\alpha\to\beta}(L) = \sum_j |U_{\alpha\, j}|^2|U_{\beta\, j}|^2 + 2\Re\left[\sum_{k>j} U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\, e^{-i\frac{(m_j^2-m_k^2)\,L}{2\,E}}\right]\,. \tag{52}$$

Now, we use the fact that $\Re[(a+i\alpha)(b+i\beta)] = a\,b-\alpha\,\beta$, the Euler formula for the exponential $e^{ix}=\cos x + i\sin x$ and the identity $\cos x = 1-2\sin^2(x/2)$ to rewrite

$$\begin{aligned} P_{\alpha\to\beta}(L) &= \sum_j |U_{\alpha\, j}|^2|U_{\beta\, j}|^2 + 2\sum_{k>j}\Re\left[U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\right] + \\ &-4\sum_{k>j}\Re\left[U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\right]\sin^2\left(\frac{(m_k^2-m_j^2)\,L}{4\,E}\right) + 2\sum_{k>j}\Im\left[U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\right]\sin\left(\frac{(m_k^2-m_j^2)\,L}{2\,E}\right)\,, \end{aligned} \tag{53}$$

where the first line at the RHS is nothing but $\delta_{\alpha\beta}$, due to the unitarity properties of the matrix U, i.e.

$$P_{\alpha\to\beta}(L) = \delta_{\alpha\beta} - 4\sum_{k>j}\Re\left[U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\right]\sin^2\left(\frac{\Delta m^2_{kj}\,L}{4\,E}\right) + 2\sum_{k>j}\Im\left[U_{\alpha\, j}U^*_{\beta\, j}U^*_{\alpha\, k}U_{\beta\, k}\right]\sin\left(\frac{\Delta m^2_{kj}\,L}{2\,E}\right)\,. \tag{54}$$

The term containing $\Im$ is the crucial term allowing for CP-violating phenomena in the neutrino oscillation phenomena (remember that if evolution of neutrinos is ruled by U, the one of antineutrinos is ruled by U^*) and was not there in the simple 2×2 case developed in the previous sub-section: vacuum CP-violation in neutrino oscillations (but not necessarily in other physical phenomena!) requires at least three generations. Also note that it averages to zero if one has too poor distance and/or energy resolution, differently from the $\Re$ one (which instead is the obvious generalization of the result derived in two generations).

Different parameterizations are possible for the matrix U (although observables such as the probability above are invariants under such reparameterizations). The most widely used consists in writing the unitary matrix U as

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix} \tag{55}$$

(here $c_{ij}=\cos\theta_{ij}$ and $s_{ij}=\sin\theta_{ij}$). This is the product of three Euler (rotation) matrices in the 1-2, 2-3 and 1-3 subspaces. For three families, neutrino flavour oscillation phenomena are governed by six independent parameters: Two mass-squared differences, $\Delta m^2_{21}\equiv m_2^2-m_1^2$ and $\Delta m^2_{32}\equiv m_3^2-m_2^2$, three mixing angles, $\theta_{12}\,,\theta_{23}\,,$ and θ_{13}, and a possible CP-violating phase $\delta_{\rm CP}$. The angles θ_{ij} can all be made to lie in the first quadrant by a redefinition of the field phases, while $0^\circ\le\delta_{\rm CP}<360^\circ$. While $m_1<m_2$, the sign of Δm^2_{32} is physical yet still unknown, and the two cases $\Delta m^2_{32}>0$ and $\Delta m^2_{32}<0$ are referred to as normal hierarchy (NH) and inverted hierarchy (IH), respectively.

(Note that other phenomena as neutrinoless double beta decay may involve two more phases, which however do not enter oscillation phenomenology).

2. A short introduction to matter effects

The previous treatment assumes that neutrinos (mass eigenstates) propagate in vacuum. It is well known from basic optics that propagation properties of photons in a medium do in general differ because of coherent interactions with the medium, giving rise to a *refractive index*. Something similar happens for neutrinos which acquire an "effective mass" while propagating in a medium that depends on medium properties such as density or temperature. The computation of these terms goes beyond the elementary level of these lectures. However, a few points are important:

- Since we are interested in the effect of these terms on the oscillation phenomenology, it is intuitively clear that what matters is the differential refractive index between different neutrino flavours (rather than the overall one).
- It is also clear that all ordinary media do contain protons, neutrons and electrons, but not heavier lepton flavours such as muons and taus. We should then expect a different behaviour between electron neutrinos (and antineutrinos) and the other species.

We will assume without proof that the evolution of a two neutrino system, written in the flavor basis, is:

$$i\frac{d}{dx}\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix}=\left(E\mathsf{I}+\frac{1}{2\,E}U\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix}U^\dagger+\Delta V\right)\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix}. \tag{56}$$

where U is the vacuum mixing matrix previously introduced (if it's the identity, there is no mixing and each state propagates independently) and

$$\Delta V=\begin{pmatrix}\sqrt{2}G_F\,n_e & 0\\ 0 & 0\end{pmatrix}, \tag{57}$$

with $\Delta V\to-\Delta V$ for antineutrinos (with all the rest unchanged) [4].

The propagating states are different from the states previously considered. They can be found by diagonalizing the previous matrix, or equivalently the following one:

$$G=\begin{pmatrix}E+(c_\theta^2m_1^2+s_\theta^2m_2^2)/2\,E+\sqrt{2}\,G_F\,n_e & s_\theta\,c_\theta\Delta m^2/2\,E\\ s_\theta\,c_\theta\Delta m^2/2\,E & E+(s_\theta^2m_1^2+c_\theta^2m_2^2)/2\,E\end{pmatrix} \tag{58}$$

A further simplification is possible if one realizes that adding any matrix proportional to the identity would not change the solution, since it only introduces an overall, common phase which is not observable. We thus perform a "cleverly chosen" subtraction,

$$G_{eff}=\mathsf{G}-\left(E+\frac{m_1^2+m_2^2}{4\,E}+\frac{\sqrt{2}}{2}\,G_F\,n_e\right)\mathsf{I}=\frac{\Delta m^2}{4\,E}\begin{pmatrix}-c_{2\theta}+A & s_{2\theta}\\ s_{2\theta} & c_{2\theta}-A\end{pmatrix}, \tag{59}$$

where $A=2\sqrt{2}E\,G_Fn_e/\Delta m^2$, and still we get the same eigenvalues of the original matrix. It is straightforward to check that

$$\frac{\Delta m^2}{4\,E}\begin{pmatrix}-c_{2\theta}+A & s_{2\theta}\\ s_{2\theta} & c_{2\theta}-A\end{pmatrix}=\frac{A\,\Delta m^2}{4\,E}\begin{pmatrix}-c_{2\theta_m} & s_{2\theta_m}\\ s_{2\theta_m} & c_{2\theta_m}\end{pmatrix} \tag{60}$$

which has the same structure of the vacuum "mixing angle", but for a new mixing angle characterized by

$$\tan(2\theta_m)=\frac{\sin(2\theta_m)}{\cos(2\theta_m)}=\frac{\sin(2\theta)}{\cos(2\theta)-A}\Longleftrightarrow\sin^2(2\theta_m)=\frac{\sin^2(2\theta)}{(A-\cos(2\theta))^2+\sin^2(2\theta)} \tag{61}$$

This has the following limiting behaviours (see Fig. 7):

- If $|A|\ll\cos(2\theta)\leq1$, there is only a marginally modified behaviour of neutrino propagation through matter. Vacuum behaviour dominates.
- If $|A|\gg\cos(2\theta)$, there is a strong suppression of the mixing. The two states are however very different from the vacuum ones, with their effective mass splitting being dominated by the matter potential (loosely speaking is like if electron and muon neutrinos almost coincide with matter eigenstates, with mass splitting given however by $A\Delta m^2=2\sqrt{2}G_Fn_e\,E$).

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[4] The form of this potential is OK for matter under ordinary conditions (and even for typical conditions in stars), but is inadequate for media such as supernova cores or the early universe.

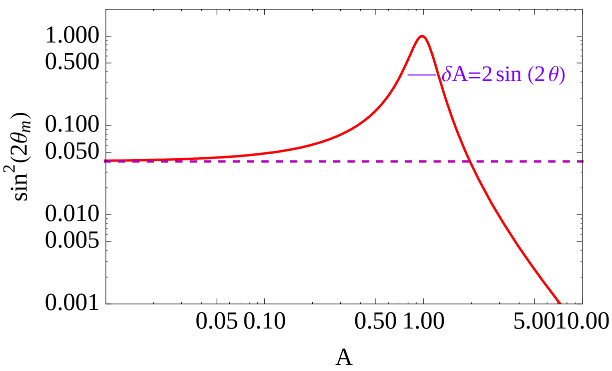

FIG. 7: The effective mixing angle in matter vs. the "matter to vacuum strength parameter" A (red, solid) compared with the vacuum mixing angle (purple, dashed). Here $\theta = 0.1$ is assumed, which also regulates the width of the peak of the solid curve, as shown.

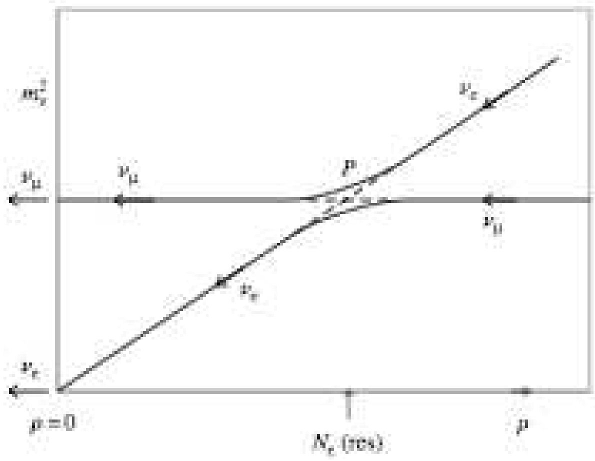

FIG. 8: The MSW effect. The neutrino mass squared is plotted against density. For "ν_μ", the mass is independent of density and is represented by the horizontal line. For "ν_e", the mass squared is proportional to the density and, if there is no flavour mixing ($\theta = 0$) is represented by the diagonal line. The two levels cross at the resonance point P. If the electron density in the dense region is greater than the resonance density at this point, the electronneutrino will be located beyond P in the upper part of the diagram. As the neutrino moves outward into regions of lower density, it eventually reaches the resonance density, and provided the density varies slowly with radius, it will move along the continuous curve and emerge from the medium as a ν_μ.

- If $A \simeq \cos(2\theta)$ (within a width roughly given by $2\sin(2\theta)$), a *resonant condition* is met, and the two states are maximally mixed even if the vacuum mixing angle is small. The density at which this resonance occurs is

$$n_e^{\rm res} = \frac{\Delta m^2}{2\,E} \tag{62}$$

- Also note that for this resonant condition to happen, also the sign of A is important (equivalently, it allows to determine the sign of Δm^2). This implies that the resonance, if any, is either happening in the neutrino or the antineutrino channel.

If a neutrino systems passes through such a resonance sufficiently slowly (in a technical sense, the scale radius mover which the density changes around the resonance must be long compared to the oscillation length), a full flavour conversion of one flavour into the other one can happen. Let's see how. In a dense region ν_e's are effectively produced in what in vacuum one would call the m_1 mass eigenstate, which is the "lightest" one in vacuum but not in the dense environment. Due to refraction, this is the most massive one in medium. If this neutrino propagates adiabatically outwards to smaller densities, it stays effectively on the "heavier" state, matching the vacuum case. But the heavier state in vacuum is m_2, the "mostly ν_μ" one. For small mixing angles, a quasi-total conversion may take place, see diagram in Fig. 8. As you will probably hear in the neutrino exp. lectures, this "matter effect" has been inferred in the study of solar neutrinos, and its analogue in the Earth crust and mantle is currently considered a tool in long baseline experiments to determine the neutrino mass hierarchy.

Further details can be found e.g. in the review [15] (OK for formalism, but outdated for the experimental part).

3. Dirac or Majorana neutrino?

We saw that the electrons emitted in a weak process obey the distribution of Eq. (29), and deduced that similarly do the neutrinos. Now, if neutrinos were massless, their momentum would be $p_\nu = E_\nu/c$, they would move at the speed of light, and their helicity would be a frame-independent (*Lorentz invariant*) quantity. All neutrinos produced in weak processes would have negative helicity, and all antineutrinos would have positive helicity.

Now, we know that neutrinos are massive. It means that we can perform a boost (change of frame) bringing us in the neutrino rest-frame, and helicity is not an invariant. The only "intrinsic" difference between ν and $\bar{\nu}$ has to do with their lepton number L (positive in the first case, negative in the latter), which in turn is related to the charged lepton they can produce. But remember that setting such a rule was an "empirical" book-keeping, there is nothing "sacred" about it as far as we know! To establish if the two are different particles or intrinsically the same, we can envisage the following *Gedanken Experiment* (see Fig. 9).

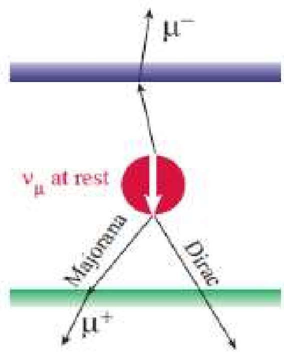

FIG. 9: Gedanken experiment to distinguish between a Dirac and a Majorana neutrino, from [16].

Suppose we have a massive ν_μ at rest with spin-down in the middle of a room. If accelerated up to relativistic energies in the up direction, when it hits the roof can produce a μ^- through a charged current (CC) interaction. If accelerated up to relativistic energies in the down direction, what happens? There are two possibilities: when it hits the floor it can produce a μ^+, or never lead to a μ^+ production (and actually no CC or NC interaction). In the former case, we would be clearly facing a violation of lepton number in a process, $\Delta L = -2$, and ν and $\bar{\nu}$ would reveal to be intrinsically the same particle. In the latter case, it means that there are other particles (technically, "right-handed neutrinos"), which are not interacting weakly (but linked to "known neutrinos" via the mass term). Neutrinos are said to be *Majorana particles* in the former case, *Dirac particles* in the latter. Testing if $L = L_e + L_\mu + L_\tau$ is a good symmetry or not is an important issue. . . and don't forget that the L_i were considered "respected" symmetries till a little more than a decade ago!

Unfortunately oscillation experiments cannot discriminate Majorana from Dirac neutrinos. The only realistic hope of experimentally discriminating Majorana from Dirac neutrino masses is based on the fact that Majorana masses violate lepton number, so that some processes otherwise forbidden may take place (e.g. they may give a signal in the future neutrinoless double beta decay ($0\nu\beta\beta$) experiments you will hear about in the Neutrino Exp. lectures).

Exercises

- The decay of Eq. (1) has a $Q - value$ of $18.6\,\mathrm{keV}$, much smaller than all the masses involved, included the electron mass $m_e\,c^2 \simeq 511\,\mathrm{keV}$. Hence non-relativistic formulae are appropriate. Use conservation of momentum to show that, in the energy conservation equation (write in rest frame of $^3\mathrm{H}$ and include rest masses!), the kinetic energy of the recoiling $^3\mathrm{He}$ can be neglected, being much smaller than the electron kinetic energy. Convince yourself that the expected electron energy spectrum should be basically monochromatic.

- Consider a hydrogen target with a density of $n = 1\,cm^{-3}$. Using Eq. (20) estimate the length ℓ a neutrino (producing an electron of energy 10 MeV) needs to cross before its "optical depth" $\tau = (\sigma\ell\,n)^{-1} = 1$. Compare it with the size of the observable universe. Alternatively, considering lead as a collection of nucleons (density of lead of about 11 g cm^{-3}, mass of the nucleon of 1.7×10^{-24} g), estimate the length of a "lead shield" necessary to intercept a majority of neutrinos (namely, determine ℓ_{Pb} in the relation $\sigma\ell_{Pb}\,n_{\mathrm{Pb}} = 1$).

- Consider the processes

$$A]\ \mu^- \ \rightarrow\ e^- + \bar{\nu}_e + \nu_\mu \tag{63}$$
$$B]\ \mu^- \ \rightarrow\ e^- + \bar{\nu}_\tau + \nu_\tau \tag{64}$$
$$C]\ \mu^- \ \rightarrow\ e^+ + \nu_e + \nu_\mu \tag{65}$$
$$D]\ \mu^- \ \rightarrow\ e^- + \nu_e + \bar{\nu}_\mu \tag{66}$$

  Are they all allowed? Which one is? Check the conservation of the lepton flavour numbers ($L_e,\ L_\mu,\ L_\tau$), as well as of their sum (total lepton number L). Is some other conservation law you are aware of violated in any of these processes?

- A generic photon polarization state can be written as a superposition of states with Horizontal and Vertical Linear Polarizations, or alternatively as a superposition of states with Right and Left Circular Polarizations. Think of neutrinos of a given flavour (i.e. at production) as being linearly polarized flavour states, while propagating neutrinos are analogous to circularly polarized states, since it is the circular polarized states that have well defined propagation characteristics such as velocity. This analogy (or some modified forms of it) can be explored further, including a simple analogical realization of the "flavour oscillation phenomenon" with lasers (see [17]).

- For two generations of *Majorana* neutrinos, the most general form for the mixing matrix is actually

$$\begin{pmatrix} \cos\theta & e^{i\,\rho}\sin\theta \\ -e^{-i\,\rho}\sin\theta & \cos\theta \end{pmatrix} \tag{67}$$

  Show that the phase ρ does not appear in the oscillation probabilities. In particular, prove that the term responsible for CP-violation in the vacuum oscillation probability formula of Eq. (54) vanishes.

PAULI'S LETTER

04/12/1930
Dear Radioactive Ladies and Gentlemen,

As the bearer of these lines, to whom I graciously ask you to listen, will explain to you in more detail, how because of the "wrong" statistics of the N and ^{6}Li nuclei and the continuous beta spectrum, I have hit upon a desperate remedy to save the "exchange theorem" of statistics and the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin 1/2 and obey the exclusion principle and which further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. The continuous beta spectrum would then become understandable by the assumption that in beta decay a neutron is emitted in addition to the electron such that the sum of the energies of the neutron and the electron is constant...

I agree that my remedy could seem incredible because one should have seen these neutrons much earlier if they really exist. But only the one who dare can win and the difficult situation, due to the continuous structure of the beta spectrum, is lighted by a remark of my honoured predecessor, Mr Debye, who told me recently in

Bruxelles: "Oh, It's well better not to think about this at all, like new taxes". From now on, every solution to the issue must be discussed. Thus, dear radioactive people, look and judge.

Unfortunately, I cannot appear in Tubingen personally since I am indispensable here in Zurich because of a ball on the night of 6/7 December. With my best regards to you, and also to Mr Back.

Your humble servant,

W. Pauli

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