A SHORT VISIT OF THE CASTLE OF PARTICLE FIELDS & SYMMETRIES
a foretaste of the quantum field theory of elementary particles and their interactions

lecture given at the
GRAduate **S**chool in **P**article and **A**stroparticle physics

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Abstract

This overview aims at providing a foretaste of a few important notions involved the formulation of the theory of elementary particles and their interactions, intended for summer student of graduate level. Emphasis will be put on ideas; technical footboards are sketched along the way whenever helpfull.

Outline

Reading would be made easier if...

you have studied, at least heard about

- Analytic Mechanics: Hamiltonian Mechanics, Lagrangian Mechanics which assumes some knowledge in functional analysis and variational calculus
- some Classical (i.e. non quantal) Field Theory, perhaps not in general but namely electro-magnetism (e-m) via Maxwell's equations; gauge invariance
- basics of Special Relativity: invariance of spacetime intervals, four-vectors, Lorentz transformations (boosts), relation between energy 3-momentum and mass
- Non Relativistic Quantum Mechanics, in particular
  * quantization of angular momentum and relation with transformations under rotations,
  * quantum particle (with or w/o spin) in an external e-m field ?
  * the quantum harmonic oscilator, and the treatment of a number of problems in terms of creators and annihilators
  * time-dependent perturbation theory, Fermi's Golden Rule ?
- perhaps an introduction to the quantization of the e-m field ?
- a zoology / panorama of elementary particles and their interactions, including the concept of Isospin ?

In preparing this lecture I have assumed so in order to anchor its content and presentation in something hopefully known.

1 Quantum Field Theory: why? what?

For quantum non relativistic systems, Quantum Mechanics provides a solid first description, with an interpretation in terms of a probability distribution of presence obeying a continuity equation. This implies 'particles conserved in times' regarding both the species and the number of representatives involved. When a higher precision is required, a refined description accounting for relativistic effects motivates relativistic quantum mechanics (RQM) - such as the Dirac equation reproducing the fine splitting spectrum of the H atom.

The physics of elementary particles and their interactions is inherently quantal and relativistic. One may wonder whether RQM, being arguably relativistic and quantal, is not just what is needed to describe it too.

1.1 Matter. Why not mere Relativistic Quantum Mechnanics?

1. RQM is *"bandy-legged"* in a number of respects
2. RQM is *not* suited to high energy physics which involves new genuine quantum relativistic phenomena
3. RQM does *not* even provide the last word for observed relativistic effects observed in low energy phenomena such as atomic spectroscopy.

Issue 1., faced when learning RQM, merely seems theoretical if not academic, yet its puzzles have to do with the fact that RQM is *apparently* addressing one-particle problems whereas relativistic quantum physics is inherently many-body. This is precisely what stresses issue 2. and, albeit indirectly, issue 3.

1.1.1 RQM is "bandy-legged"

RQM suffers from a number of puzzles and paradoxes, let us mention only one, bad enough:

negative energy solutions: the spectrum is not bounded below!
Consider the dispersion relation of plane wave solutions for free particles. The non relativistic relation $E = \vec{p}^2/(2m)$ linear in E implies that E is > 0. On the contrary the non relativistic dispersion relation $E^2 = \vec{p}^2c^2 + m^2c^4$ is *quadratic* in E and allows two solutions of opposite signs: $E_\pm = \pm\,(\vec{p}^2c^2 + m^2c^4)^{1/2}$: there is an infinite family of relativistic wave solutions with $E_- < 0$ unbounded below: the hamiltonian has *no* fundamentral state, a disaster! As a *last resort* Dirac proposed to fill this infinite sea of < 0 levels relying on Pauli's exclusion principle (single occupancy of fermionic energy levels), interpreting holes in the sea as *anti-particles*. He thereby dealt with *an infinite number of particles* in a setup minded to describe only one at a time... Worse, the relativistic wave eqn. for spin 0 particles - the Klein-Gordon eqn. - has the same "trouble", which in this case however can *not* be knocked-up in this ad hoc manner as spin 0 particles are bosons not obeying Pauli's principle...

The concept of anti-particle was indeed a successful theoretical prediction of RQM in 1930, confirmed 2 years later by the discovery of the positron in cosmic ray events, but the theoretical setup has to be recasted completely.

1.1.2 High Energy Physics involves genuine quantum & relativistic phenomena

The relativistic equivalence between energy and mass in the quantum framework implies processes in which both the *arithmetic* number of particles and even the *nature* of the species involved can *change discontinuously*:

1. **extra particles can be produced** in scattering processes, "embodying" part of the incoming kinetic energy;
2. **incoming particles may co-annihilate and turn into other particle species**
3. **unstable particles decay into products not being their subconstituants**.

For example, at the LHC, protons collide at 8 TeV in the c.m.s. frame. One subconstituant of one proton named gluon carrying a fraction $\sim 10^{-2}$ of its four-momentum coannnihilates with a similar one from the other proton, to form a Higgs boson of mass 125 GeV$/c^2$. The Higgs boson quickly decays into a Z boson and a highly energetic elecron-positron pair. The $\sim$ 91 GeV$/c^2$ heavy Z boson in turn quickly decays into a muon-antimuon pair. The $\sim$ 105 MeV$/c^2$ (anti)muon, unstable yet much lighter than the decaying Z, are produced so boosted in the lab frame that they will not have time to decay before they are eventually detected and can be considered stable.

The description of such a process and the calculation of its probability rate calls for a theoretical setup creating particles here, propagating from here to there and annihilating there. Collections of creators and annihilators distributed in spacetime: here are quantum fields!

1.1.3 Some low energy phenomena go beyond RQM

Those phenomena are refered to as "Radiative corrections". Here is one example - the archetype which became the cornerstone of the ackowledgement of Quantum Electrodynamics, more generally Quantum Field Theory as the framework to describe particle physics:

Anomalous magnetic dipole moments of leptons: $g \neq 2$
Landé's g-factor of elementary fermions such as the electron, or the muon relates their magnetic dipole moment $\vec{\mu}$ and their spin $\vec{S}$ by

$$\vec{\mu} = g \, \frac{e}{2m} \vec{S}$$

A shown below, g is predicted to be exactly 2 by Quantum Mechanics (QM), whether Relativistic (Dirac eqn.) or Non Relativistic (Pauli-Schrödinger eqn., to which the former reduces at low energy) assuming the so-called "minimal coupling prescription". In classical mechanics, this prescription replacing $\vec{p}$ by $\vec{p} - e\,\vec{A}(\vec{x})$ in the kinetic part of the Hamiltonian of a free charged particle reproduces the Lorentz force $\vec{F} = e\,\vec{v} \wedge \vec{B}$ caused by the magnetic field $\vec{B} = \vec{\nabla} \wedge \vec{A}$. In QM, the corresponding substitution is

$$\frac{\hbar}{i} \vec{\nabla} \quad \rightarrow \quad \frac{\hbar}{i} \vec{\nabla} - e\,\vec{A} \tag{1.1}$$

in the kinetic part of the Hamiltonian. A low energy spin 1/2 particle is described by a wave function consisting in a column vector with two complex components, named spinor, on which acts the Pauli Hamiltonian including the so-called "minimal coupling" term:

$$H_{m.c.} = -\frac{\hbar^2}{2m}\left(\vec{\sigma}\cdot\left(\vec{\nabla} - i\,\frac{e}{\hbar}\vec{A}\right)\right)^2 \tag{1.2}$$

involving the Pauli matrices defined by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;,\; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;,\; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \vec{\sigma}\cdot\vec{P} \equiv \sum_{j=1}^{3} \sigma_j\, P_j$$

Using the identity fulfilled by Pauli matrices $\sigma_i\sigma_j = \delta_{ij}\,\mathbb{1}_2 + i\,\epsilon_{ijk}\,\sigma_k$, $H_{m.c.}$ involves a term

$$H_{mag} = -\left(\frac{e}{m}\right)\left(\frac{\hbar}{2}\,\sigma_k\right)\epsilon_{kij}\,\left(\nabla_i\,A_j + A_i\,\nabla_j\right) \tag{1.3}$$

where the action of the last factor of eqn. (1.3) on a wave function $\psi(x)$ is equal to the left multiplication by $\epsilon_{kij}\,(\nabla_i\,A_j) = B_k$ so that $H_{m.c.}$ reads:

$$H_{m.c.} = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - i\,\frac{e}{\hbar c}\vec{A}\right)^2\,\mathbb{1}_2 - \underbrace{2\,\frac{e}{2m}}\vec{S}\cdot\vec{B} + \cdots \tag{1.4}$$

Early experiments in the late 1940's however measured a small but significant departure from 2: $(g-2)/2 \simeq 1.16\,10^{-3}$. The interaction between charged spinning particles and fluctuations of the quantum e.m. field amounts to an "extra" magnetic dipole moment contribution $(g-2)/2 \simeq \alpha/(2\pi)$, first computed by Schwinger in 1948. At a higher level of experimental precision and corresponding theoretical accuracy $\lesssim 10^{-5} \sim \mathcal{O}((\alpha/\pi)^2)$, the sensitivity of $g-2$ of the elctron and muon to quantum fluctuations of matter fields make them depart from each other. The $(g-2)_{electron}$ has been both measured and computed at accuracy 10^{-13}, one of the finest tests of QED. The $(g-2)_{muon}$ has been both measured and computed at accuracy 10^{-10}. Tensions in the comparison th. vs. exp. at this accuracy might suggest effects of possible "new physics" not accounted in the calculation.

Particles coupling to vacuum fluctuations of fields induce so-called **virtual** effects i.e. occuring w/o any actual participation of the particles associated with the fluctuating fields. These virtual effects are potentially measurable in precision experiments, probing the quantum relativistic world of elementary particles in a way complementary to the so-called "direct searches" tentatively observing new particles. Beside the $(g-2)_{muon}$ example, precision measurements at LEP were interpreted in the Standard Model framework in terms of radiative corrections sensitive to the top quark mass. The latter was correctly "sensed" around 170 GeV, before the actual observation of the top at the Fermilab Tevatron, whereas the LEP operating conditions did not allow to produce top quarks.

1.2 Field Theory: a sketchy introduction

Herebelow is sketched a formal dictionary to leap from Analytic Mechanics to Classical Field Theory seen as "Analytic Mechanics of infinitely many degrees of freedom". It's intended to help, but if you instead find it fearsome, skip it without damage! The notion of quantum field is then introduced using the principle of correspondence.

1.2.1 A Dictionnary for 'Analytic Mechanics → Classical Field Theory'

The 3-space dependence of the field is singled out from the time dependence and the 3-space dependence is treated as a continuous index.

	finite sums $\sum_{j=1}^{n}$	$\rightarrow$	integral over 3-space $\int d^3x$
	trajectory $q_j(t)$	$\rightarrow$	field configuration $\phi(t,\vec{x})$
Lagrangian density:			$\mathcal{L}\left(\phi(t,\vec{x}), \partial_t\phi(t,\vec{x}), \vec{\nabla}\phi(t,\vec{x})\right)$
Lagrangian:	$L\left(\{q_i(t), \dot{q}_i(t)\}\right)$	$\rightarrow$	$L = \int d^3x\, \mathcal{L}$
action:	$\mathcal{S} = \int dt\, L\left(\{q_i(t), \dot{q}_i(t)\}\right)$	$\rightarrow$	$\mathcal{S} = \int d\, L = \int d^4x\, \mathcal{L}$
Euler-Lagrange eqn.:	$\frac{\delta \mathcal{S}}{\delta q_j(t)} = 0$	$\rightarrow$	$\frac{\delta \mathcal{S}}{\delta \phi(x)} = 0$
	i.e.		*i.e.*
	$\frac{\partial L}{\partial q_j(t)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j(t)}\right) = 0$	$\rightarrow$	$\frac{\partial L}{\partial \phi(x)} - \partial_t\left(\frac{\partial L}{\partial(\partial_t\phi(x))}\right) - \vec{\nabla}\cdot\left(\frac{\partial L}{\partial(\vec{\nabla}\phi(x))}\right) = 0$
congugate momentum:	$p^j(t) = \frac{\delta \mathcal{S}}{\delta q_j(t)} = \frac{\partial L}{\partial q_j(t)}$	$\rightarrow$	$\pi(t,\vec{y}) = \frac{\delta \mathcal{S}}{\delta \partial_t\phi(y)} = \frac{\partial \mathcal{L}}{\partial(\partial_t\phi(y))}$
Hamiltonian density:			$\mathcal{H} = \pi(x)\left(\partial_t\phi(x)\right) - \mathcal{L}$
Hamiltonian:	$H = \left[\sum_j p^j(t)\, q_j(t)\right] - L$	$\rightarrow$	$H = \int d^3x\, \mathcal{H}$
from partial to functional derivatives:	$\frac{\partial q_i}{\partial q_j} = \frac{\partial q^i}{\partial p^j} = \delta_{ij}$	$\rightarrow$	$\frac{\delta\phi(t,\vec{x})}{\delta\phi(t,\vec{y})} = \frac{\delta\pi(t,\vec{x})}{\delta\pi(t,\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y})$
	$\frac{\partial q_i}{\partial p^j} = 0,\ \frac{\partial p^i}{\partial q_j} = 0$	$\rightarrow$	$\frac{\delta\phi(t,\vec{x})}{\delta\pi(t,\vec{y})} = 0,\ \frac{\delta\pi(t,\vec{x})}{\delta\phi(t,\vec{y})} = 0$
Poisson Bracket $\{A,B\}$:	$\sum_j\left[\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p^j} - \frac{\partial A}{\partial p^j}\frac{\partial B}{\partial q_j}\right]$	$\rightarrow$	$\int d^3x\left[\frac{\delta A}{\delta\phi(x)}\frac{\delta B}{\delta\pi(x)} - \frac{\delta A}{\delta\pi(x)}\frac{\delta B}{\delta\phi(x)}\right]$
canonical conjugation:	$\left\{q_j(t), p^j(t)\right\} = \delta_{ij}$	$\rightarrow$	$\{\phi(,\vec{x}), \pi(,\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})$
Hamilton eqns.:	$\dot{q}_i = -\{H, q_i(t)\}$	$\rightarrow$	$\partial_t\phi(y) = -\{H, \pi(y)\}$
	$= \frac{\partial H}{\partial p^i(t)}$		$= \frac{\delta H}{\delta\pi(y)} = \frac{\partial\mathcal{H}}{\partial\pi(y)}$
	$\dot{p}^i = -\left\{H, p^i(t)\right\}$	$\rightarrow$	$\partial_t\pi(y) = -\{H, \pi(y)\}$
	$= -\frac{\partial H}{\partial q_i(t)}$		$= -\frac{\delta H}{\delta\phi(y)} = -\frac{\partial\mathcal{H}}{\partial\phi(y)}$

1.2.2 The picture which we have in mind

To keep the picture the simplest possible, we start with free, i.e. non interacting fields: we have ultimately in mind to couple such fields together according to a perturbative approach inspired by a collisional picture:

1. incoming particles separately prepared in a remote past, idealy in a monokinetic state or in wave packets sharpkly peaked in momentum space, freely fly towards each other

2. they interact in a finite region of space during a limited time, hopefully weakly enough for the interaction to be described by perturbation theory and compute experimentally measured quantities

3. outgoing particles freely escape away from each other in a remote future when are eventually detected.

This intuitive approach motivates the formulation of a field-theoretical perturbation theory (elaborating on the time-dependent perturbation theory which you may have learnt in QM) named the interaction picture. Perturbation theory means that the calculations of physical quantities will rely on a Taylor expansion in integer powers of coupling parameters. The latter shall be hopefully small enough to make the expansion meaningfull and effective in practice. To be fair, the actual implementation of this programme raises a number of complications, both conceptual and technical... skipped here.

1.2.3 From classical to quantum fields

Let us start with relativistic, classical i.e. non quantum fields. To peel off all the complications linked with spin issues, we consider a scalar i.e. spinless field $\phi_{clas}(x)$. It is a wave packet solution of the equation of motion (Euler-Lagrange eqn.) derived via a principle of least action from an action given by a free Lagrangian density. $\phi_{clas}(x)$ decomposes into its Fourier modes of positive and negative energies:

$$\phi_{clas}(x) \;=\; \sum_{\vec{p}} \left[\beta_+(\vec{p})\, e^{-ip.x} \,+\, \beta_-(\vec{p})\, e^{+ip.x}\right] \tag{1.5}$$

The *numbers* $\beta_+(\vec{p})$ and $\beta_-(\vec{p})$ (in general complex) are the amplitude coefficients of the Fourier modes. The field is supposed complex-valued so as to describe charged particles; if real we have $\beta_-(\vec{p}) = \beta_+^*(\vec{p})$. The Fourier phases $e^{\mp i\,p.x}$ involve the relativsically invariant combination $p.x \equiv E(p)t - \vec{p}.\vec{x}$. The shorthand "$\sum_{\vec{p}}$" stands for an integral over 3-momentum implying $E(p) = (p^2c^2 + m^2c^4)^{1/2}$. Let us open one parenthesis here:

`(Parenthesis on High Energy Physics conventions for units & dimensions`
In Special Relativity you may be used to note positions with four-vectors $(ct, \vec{x})$ and four-momenta $(E/c, \vec{p})$ for energy-momenta. One customarily introduces $x^0 = ct$ and $p^0 = E/c$ so as to treat on equal footing time and length on one hand, energy and momentum on the other hand. A quick way to handle these relativistic notational practices is to formally "put $c = 1$" so that c disappears from all eqns. Thereby, the dimensions of times and lengths become equal; the three dimensions of masses, energies and momenta become all equal. The relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$ now reads: $E^2 = p^2 + m^2$.

Likewise, in Quantum mechanics, it is convenient to identify wave numbers k and momenta $p = \hbar\, k$, resp. frequencies ω and energies $E = \hbar\, \omega$, by "putting $\hbar = 1$". Action and angular momentum (having same dimension as $\hbar$) become "pure numbers", lengths have same dimension as inverse of momenta, times have same dimension as inverse of energies. Combining "$\hbar = c = 1$" amounts to a common dimension of masses, energies & momenta inverse of the common dimension of times & lengths. Appropriate powers of $\hbar$ and c can be restored by a careful dimensional analysis - `end of parenthesis)`

The principle of correspondence Classical Field → Quantum Field is formally the same as for Hamiltonian point Mechanics → Quantum Mechanics in the Heisenberg picture. It imposes an equal time commutation condition for the quantum fields and conjugate moments. For bosons we get:

$$\left[\phi(x^0,\vec{x}),\pi(x^0,\vec{y})\right] \quad = \quad i\,\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.6}$$

with all other equal-time commutators vanishing. The Fourier expansion of the quantum field $\phi(x)$ is obtained by replacing the numbers $\beta_{\pm}(\vec{p})$ in eqn. (1.5) by *operators*. Let us temporarily note these operators $b_{\pm}(\vec{p})$ untill they are properly intepreted:

$$\phi(x) \quad = \quad \sum_{\vec{p}} \left[b_+(\vec{p})\, e^{-ip.x} \; + \; b_-(\vec{p})\, e^{+ip.x}\right] \tag{1.7}$$

Condition (1.6) implies the following commutation relations (positive normalization coefficients irrelevant for the point which we make here are omitted):

$$\left[b_+(\vec{p}), b_+^{\dagger}(\vec{p}')\right] = \left[b_-^{\dagger}(\vec{p}), b_-(\vec{p}')\right] \quad = \quad \delta^{(3)}(\vec{p}-\vec{p}') \tag{1.8}$$

$$\text{all other commutators vanishing} \tag{1.9}$$

Relations (1.8) tell that the $b_+(\vec{p})$ and $b_-(\vec{p})$ shall be interpreted as annihilators of particles and creators antiparticles respectively: so let us renote them $b_+(\vec{p}) \equiv b(\vec{p})$ and $b_-(\vec{p}) \equiv d^{\dagger}(\vec{p})$, while the hermician conjugate field $\phi^{\dagger}(x)$ is decomposed onto $d(\vec{p})$ and $b^{\dagger}(\vec{p})$ putting the treatment of particles and antiparticles on an equal footing. The vacuum $|\,0\,\rangle$ is defined as the unique state ahhihilated by all the b and d, it is the state of lowest energy. The Fock space is spanned by the states obtained by the iterative action of $b^{\dagger}$ and $d^{\dagger}$ on $|\,0\,\rangle$. This interpretation of the operators introduced in decomposition (1.7) is the one which corresponds to a Fock space of states with positive define hermician norm squared i.e. a Hilbert space, and is such that the Hamiltonian acting on this space is bounded below.

You may have met creators and annihilators in your QM course already: when studying the quantum harmonic oscillator. Indeed a free field may be seen as an infinite collection of harmonic oscillators. One can elaborate a little bit on this analogy, as it will give a persepective usefull in wing 3. Let us consider an harmonic oscilator with "formal unit mass" and frequency m^2 noted in this unusual way on purpose. the Hamitonian and corresponding Lagrangian of the harmonic oscillator read:

$$H = \frac{1}{2}\,p^2 + \frac{1}{2}\,m^2\,x^2 \quad , \quad L = p\dot{x} - H = \frac{1}{2}\,\dot{x}^2 - \frac{1}{2}\,m^2\,x^2 \tag{1.10}$$

Likewise the Lagrangian density of a relativistic free scalar field reads:

$$\mathcal{L} \;=\; = \frac{1}{2}\left[(\partial\phi)^2 \,-\, \left(\vec{\nabla}\,\phi\right)^2\right] - \frac{1}{2} m^2\,\phi^2 \tag{1.11}$$

in which the analogy between the kinetic terms on both sides on one hand, as well as between the quadratic potential term on the oscillator side $\leftrightarrow$ the mass term on the field theory side the other hand, is obvious. A set of interacting fields $\{\phi_j\}$ will formally correspond to oscillators x_j coupled together by adding non anaharmonic terms in the potential $V(\{x_j\})$. In this analogy, the mass spectrum of the fields $\{\phi_j\}$ on the field theory side correspond to the vibration eigen-modes of small amplitude around the classical equilibrium position given by the minimum of the potential. These eigen-frequencies are the eigenvalues of the (matrix of) second derivatives of the potential $V(\{x_i\})$:

$$M^2_{jk} = \left[\; \frac{\partial^2 V}{\partial x_j\,\partial x_k} \;\right]_{V\,=\,V_{min}} \tag{1.12}$$

computed at the minimum of the potential V. As currently done for anharmonic terms in QM, interactions among the fields will be treated perturbatively as sketched in piece 4 and their effects will be computed as a Taylor expansion in powers of the corresponding coupling parameters. Last, the minimum may be continuously degenerate, and may correspond to a non zero value for the mean positions x_0 of some of the guys w.r.t. the origin. We will come back to this possibility, very important on the field theory side, in section 3.

Fermion fields require instead that quantization be imposed by means of equal time *anti-commutors*, which in turn lead to the replacement of commutators by anticommutators in relation (1.8). At variance with Dirac's picture of the filled infinite negative sea, particles and antiparticles are now put on equal footing considering the symmetry of roles of $\phi(x)$ and $\phi^\dagger$, whether bosons or fermions. For example, (let aside notational details), whereas the energy of the plane wave solution (1.5) for a spin 1/2 field in RQM is found equal to

$$\mathcal{E} \;=\; \sum_{\vec{p}} E(\vec{p})\left[\beta^*_+(\vec{p})\,\beta_+(\vec{p}) \,-\, \beta_-(\vec{p})\,\beta^*_-(\vec{p})\right]$$

which has indefinite sign, the Hamiltonian of the quantum field theory (QFT) is shown to take the form

$$H \;=\; \sum_{\vec{p}} E(\vec{p})\left[b^\dagger(\vec{p})\,b(\vec{p}) \,+\, d^\dagger(\vec{p})\,d(\vec{p})\right]$$

it is a nonnegative operator. On the contrary, the integral over the positive definite space density in RQM

$$\mathcal{N} = \int d^3x\,\psi^\dagger(x)\psi(x) = \sum_{\vec{p},s}\left[\beta^*_+(\vec{p})\,\beta_+(\vec{p}) \,+\, \beta_-(\vec{p})\,\beta^*_-(\vec{p})\right]$$

is replaced in QFT by the counting operator

$$N \;=\; \sum_{\vec{p}}\left[b^\dagger(\vec{p})\,b(\vec{p}) \,-\, d^\dagger(\vec{p})\,d(\vec{p})\right] \tag{1.13}$$

N is algebraically counting $+1$ for each particle and -1 for each anti-particle: this quantum field theory handles an arbitrary number of particles and antiparticles.

2 Symmetries, conservation laws and labelling of states

Symmetries play a key role in high energy physics. In brief, a system is said to possess a symmetry when some transformation of this system lets its equations of motion (e.o.m.) unchanged. Actually one speaks of a symmetry *whether* one considers a transformation which *acts on the system itself*, or when one considers a *transformation of the description* of this system e.g. by changing observers frames. The first case tells about a *feature of the system itself*, the second case instead states a *principle of relativity of some kind*. This semantic sloppiness is partly due to the fact that the same mathematical notions are involved in both cases: group transformations. In particular, continuous symmetries correspond to continuous groups - named Lie groups - such as the rotation group, or so-called "internal symmetries" as opposed to space- or spacetime transfomations. Typical examples of "internal symmetries" are phase reparametrizations of complex charged fields, or Isospin transformations as met in Nuclear and Hadronic physics.

Figure 1: Chambord, illustrating the Castle of Symmetries...

Continuous symmetries, whether of spacetime or "internal", are of utmost importance as

2.1 Continuous symmetries imply conservation laws!

This very general result holds for relativistic or non relativistic systems, for field theory as well as point particles - for which you must have learnt several important examples already:

invariace under time translations	$\Rightarrow$	total energy is conserved
invariance under space translations	$\Rightarrow$	total momentum is conserved
invariance under rotations	$\Rightarrow$	total angular momentum is conserved

2.1.1 Field theoretical case: Noether's theorem

Within the framework of field theory given by a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ depending only on the field ϕ and first derivativews w.r.t. time and space $\partial_\mu \phi$, this property stated by **Noether's theorem** in a *local* form: to each continuous symmetry corresponds a conservation law in the form of a continuity eqn. for a (or a collection of) spatial density (-ies) and current(s) - which in the relativiistic case form so-called four-current vectors, of the type

$$\frac{\partial\rho}{\partial_t} + \nabla_k j^k = 0 \ \text{ i.e } \ \partial_\mu j^\mu = 0, \ \ j^\mu = (\rho, \vec{j})$$

ρ and $\vec{j}$ are expressed in terms of the fields satisfying the e.o.m. and the derivatives of these fields. By integrating the "spatial "density" ρ over 3-space volume (and *assuming* that the fields *decrease fast enough* as spatial ∞ so that $\vec{j}(t,\vec{x}) \to 0$ fast enough as spatial ∞ too...) one defines a time independent i.e. conserved "generalized charge":

$$\frac{d}{dt}\int d^3x\, \rho(t,\vec{x}) = -\int d^3x\, \nabla.\vec{j}(t,\vec{x}) \ = \ -\int_{\|\vec{x}\|\to\infty} d^2\vec{\sigma}.\vec{j}(t,\vec{x}) = 0 \tag{2.14}$$

where the last equality uses Green's formula saying that the div of a vector field in a space volume is equal to the flux of this vector field accross the surface forming the boundary of this volume. Noether's argument does even more: if the continuous transformation considered is *not* a symmetry, Noether's argument allows to construct a density & current whose obey a continuity eqn. with a non vanishing r.h.s., also provided by the constructive variational calculus of the argument and which tell what is the obstruction to a symmetry.

A toy model

Let us illustrate these ideas and results on a toy-model quite familiar to you: "non relativistic wave quantum mechanics" of a point particle, albeit from a novel point of view. Indeed the quantum wave function obeying the Schrödigner eqn. can be technically seen as a non relativistic clasical field, and the Schrödinger eqn. can be seen as deriving from a least action principle from the following Lagrangian density:

$$S = \int_{t_i}^{t_f} dt\, d^3x\ \mathcal{L}(\psi, \psi^*, \text{and their first time- and space- derivatives}) \tag{2.15}$$

$$\mathcal{L} = \frac{i}{2}\Big[\psi^*(t,\vec{x})\ (\partial_t\psi(t,\vec{x})) - (\partial_t\psi^*(t,\vec{x}))\ \psi(t,\vec{x})\Big] - \frac{1}{2m}\vec{\nabla}\psi^*(t,\vec{x}).\vec{\nabla}\psi(t,\vec{x}) - V(\vec{x}))\,\psi^*(t,\vec{x})\psi(t,\vec{x}) \tag{2.16}$$

As a warm-up, let us derive the Euler-Lagrange eqn. for $\psi(x)$ by variational calculus w.r.t $\psi^*(t,\vec{x})$ treated as independent field. Following dictionary 1.2.1:

$$\frac{\partial\mathcal{L}}{\partial\psi^*(t,\vec{x})} = \frac{i}{2}\ (\partial_t\psi(t,\vec{x})) \quad - \quad V(\vec{x}))\,\psi(t,\vec{x})$$

$$\frac{\partial\mathcal{L}}{\partial({}_t\psi^*(t,\vec{x}))} = -\frac{i}{2}\,\psi(t,\vec{x}) \quad , \quad \frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\psi^*(t,\vec{x}))} = \frac{1}{2m}\,\vec{\nabla}\psi(t,\vec{x})$$

Hence the E.L. eqn.:

$$i\,(\partial_t\psi(t,\vec{x})) - \frac{1}{2m}\,\vec{\nabla}^2\,\psi(t,\vec{x}) - V(\vec{x}))\,\psi(t,\vec{x}) = 0 \tag{2.17}$$

which is nothing but the Schrödinger eqn.! The warm-up is usefull to solve the following.

We can now examine how the system "complex field ϕ coupled with the exterior potential $V(\vec{x})$" transforms under time-translations, space-translations, rotations, global phase reparametrizations, with a fresh eye of Noether's argument: get the associated Noether's currents and study whether or not they lead to conserved "charges" and why. This is left as an exercise, which will have you rediscover result from your QM course in a novel way! As a handle, here are the answers:

- For time translations $t \to t + \delta t$: $\delta_t \, \phi = \delta t \, \partial_t \phi$. The "charge" is the Hamiltonian, i.e. total total energy. It is conserved as V is assumed t-independent.

- For space translations $\vec{x} \to \vec{x} + \delta\vec{a}$: $\delta_{\vec{a}} \, \phi = \delta\vec{a}.\vec{\nabla}\phi$. Noether's argument leads to three currents, the corresponding "charges" are the three components of momentum. A momentum component along a given direction is conserved i.e. obeys a continuity equation with vanishing r.h.s. *if and only if* $\vec{\nabla}V = 0$ along this direction. $\vec{\nabla}\, V(\vec{x}) \neq 0$ classicaly means that a force derives from the potential, which in the corresponding classical case implies that the momentum is not conserved,cf. Newton's 1^{st} Fondamental Law of Dynamics in classical mechanics: $d\vec{p}/dt = \vec{F} = -\vec{\nabla}V$. Noether's argument applied here... Ehrenfest's theorem! i.e. a quantum counterpart with quantum observables (operators) averaged over the wave function!

- Likewise for rotations: the "charges" are the components of angular momentum. In the spinless case considered they are purely orbital (of torque type, induced by the so-called transport term in the variation of ψ). Would we consider the Pauli-Schrödinger eqn. e.g. for spin 1/2 instead of the spinless one, we would get spin contributions in the "angular momentum charges". Besides, all "angular momentum currents" are conserved iff V is spherically symmetric i.e. depends only on $\|\vec{x}\|$. In case the potential has only some reduced symmetry (less than spherical, e.g. cyclindric) only *some* components of the angular momentum conserved, not all of them (cf. Newton's 2^{nd} Fondamental Law of Dynamics in classical mechanics $d\vec{J}/dt = \vec{\mathcal{M}}$). Noether's argument provides again Eherenfest theorem for the correspondance with the Newton's 2^{nd} Fondamental Law of Dynamics

- The internal symmetry here = global phase shifts of the field $\psi(t, \vec{x})$: $\delta_\omega \, \psi(x) = i\delta\omega \, \phi$. The continuity eqn. provided by Noether's argument here is nothing but... The continuity eqn. for the probability density of presence at position $\vec{x}$ at time t and coresponding probability current! This can be reinterpreted as a conservation law of particle species, or of electric charge for this 1-particle case.

Conclusion and lesson: symmetries imply conserved "charges" in a generalized sense. In the quantum theory, those "generalized charges" are operators: their eigenvalues provide quantum numbers to label particle states.

2.2 Symmetries and labelling of particles states

You have hopefully met this issue already, in a the context of the quantization of angular momentum in Quantum Mechanics. The algebra of commutation relations $[J_a, J_b] = i\epsilon_{abc}J_c$ obeyed by the components J_a $a = 1, 2, 3$ of the angular momentum operator, is nothing but the so-called Lie algebra of the rotation group in 3-dimensional space, because the components of the angular momentum operator are the generators of space rotations. The quantization of angular momentum identifies with the construction and classification of all the ways rotations can act on spinning quantum systems, which is named the *unitary representations* - unitary because the J_c are hermician. The representations are labelled according to the spin quantum number j related to the eigenvalue $j(j+1)$ of the quadratic combination $\vec{J}^2 = \sum_{a=1}^{3} J_a J^a$. The labelling of the states forming an eigenbasis in the eigenspace of spin j are labelled by one of the J_a, say J_3.

The labelling of particle states with relaltivistic quantum numbers consists in a relativistic generalization of the quantization of angular momentum, the rotation group in 3-dim space being replaced by a larger group named the Poincaré group containing the spacetime translations, the rotations and the Lorentz boosts. There are two physical sorts:

2.2.1 The Poincaré group and the kinematical quantum numbers of particles

Let aside $p = 0$, labelling the vacuum, there are two interesting types for physics.

$m^2 > 0$:
corresponds to massive particles. The quantum numbers provide the same kinematic labelling here as for a non relativistic spinning system. No surprise: a massive state possesses a rest frame: once bosted back into its rest frame, a massive particle... rests, non relativistic!

$m^2 = 0, p \neq 0$:
there is no rest frame for p, so this new type has *no equivalent* in nonrelativistic spinning systems. The direction of motion gives a priviledged direction, the only information on spin is the spin projection along the direction of the 3-momentum $\vec{p}$ named helicity. In the physically relevant cases, the representations are *one-dimensional* and labelled by helicity taking one half integer value s, either positive or negative when non zero. When $s = 0$ there is only one representation, invariant under parity (the transformation which reverses the orientation of space). When $s \neq 0$ there are two distinct representations with opposite helicities (notice that boosts can't flip the sign of helicity in the massless case) which are exchanged under parity: two-dimensional representations accounting for parity are thus made of the direct sum of the two. Massless states are labelled by $m = 0$, four-momentum p and helicity s.

NB: $m^2 < 0$ corresponds to the so-called tachyons. Forget about tachyons.

Let us stress in particular the **following important difference**:

> **massive spin** $s = 1$ particles like W and Z bosons have $2s+1 = 3$ degrees of freedom i.e. polarization states like a non relativistic spin 1,
>
> **massless helicity** $s \pm 1$ particles like photons have only $1 + 1 = 2$, cf. the circular left and and right polarizations of light.

⇒ **To make a massless vector boson massive requires not only to give it a mass, but also requires to provide one more polarization state** / degree of freedom, which has to come from somewhere... Keep it in mind about the so-called Higgs mechanism!

3 Symmetries & interactions

Continous symmetries do not only provide quantum numbers labelling particle states. Local or "gauge symmetries" *provide the modelling* of their interactions! The wording "gauge symmetries" shall be understood as the statement of some kind of "principle of relativy" named gauge invariance, which may be stated as follows:

At any spacetime point, there does not exist any priviledged origin of phases, nor any priviledged reference basis in "internal space" to get one's bearings

3.1 Local symmetry implies interaction!

We frame the argument in non relativistic wave mechanics, familiar to you and good enough to present the idea. The free Schrödinger eqn. is invariant under *global* phase reparametrization $\psi(t,\vec{x}) \rightarrow e^{i\omega}\,\psi(t,\vec{x})$, $\omega = cst$. Let us undertake to infer a model whose wave eqn. be invariant now under *local* i.e. space & time -dependent transormations

$$\psi(x) \rightarrow f(x)\,\psi(x), \quad f(x) = e^{i\,e\,q\,\omega(x)} \tag{3.18}$$

in accordance with the above principle of "gauge invariance", and interprete it. In eqn. (3.18) have been introduced two parameters e and q, to be further interpreted as a coupling parameter and a charge respectivelly, in a somewhat artifial way at this stage. Actually one does not wish the field $\psi(t,\vec{x})$ alone to transform covariantly under (3.18), one shall require that time and space *derivatives* of the field do as well. This is obviously not the case for the usual derivatives $\partial_t\phi(t,\vec{x})$, $\vec{\nabla}\phi(t,\vec{x})$ as

$$\partial_t\psi \quad\rightarrow\quad \partial_t\,(\,f\,\psi\,) = f\ (\partial_t\psi) + (i\,e\,q\,(\partial_t\,\omega))\,\psi \tag{3.19}$$

$$\vec{\nabla}\psi \quad\rightarrow\quad \vec{\nabla}\,(\,f\,\phi\,) = f\,\left(\vec{\nabla}\psi\right) + \left(i\,e\,q\,(\vec{\nabla}\,\omega)\right)\psi \tag{3.20}$$

Let us think at what we are doing: a derivative consists in comparing the values of the field at infinitesimaly nearby points. If one still gets something meaningful despite the arbitrary phase changes here and there, a *messenger* is required to compensate for the arbitrary phase change in order to perfom the comparison: the desired theoretical setup cannot be a free one, we are implicitely *requiring* the existence of an *interaction* involving this messenger! The simplest way to practically proceed is to modify the usual derivative by adding a multiplicative term involving the new messenger field(s) $A_t(t,\vec{x})$, $\vec{A}(t,\vec{x})$ (have in mind that in a special relativistic framework the two altogether will become components of a single four-vector-valued field), and replacing

$$\partial_t \rightarrow D_t \equiv \partial_t - i\,e\,q\,A_t \ \ , \ \ \vec{\nabla} \rightarrow \vec{D} \equiv \vec{\nabla} - i\,e\,q\,\vec{A} \tag{3.21}$$

concomitantly imposing a transformation law on the messenger field(s) which compensates the extra terms involving the derivatives of the arbitrrary phase shift which appeared in in eqns. (3.19), (3.20), so that the derivatives of ψ defined by eqn. (3.21) transform covariantly in the following sense:

$$D_t\psi \quad\longrightarrow\quad D'_t\,(f\,\psi) \;=\; f\,D_t\psi(x) \tag{3.22}$$

$$\vec{D}\psi \quad\longrightarrow\quad \vec{D}'\,(f\,\psi) \;=\; f\,\vec{D}\psi(x) \tag{3.23}$$

where we noted $D'_t = \partial_t - i\,e\,q\,A'_t$, $\vec{D}' = \vec{\nabla} - i\,e\,q\,\vec{A}'$ for the observer 'prime'. The transformation law on the messenger field(s) is infered to be:

$$A_t \longrightarrow A'_t = A_t + \partial_t \omega \;\;,\;\; \vec{A} \longrightarrow \vec{A}' = \vec{A} + \vec{\nabla}\omega \tag{3.24}$$

But wait! this is not new: the modified derivative is namely the minimal coupling prescription in electromagnetism seen in subsubsec. 1.1.3, the messenger field transforming as a gradient of some arbitrary function corresponds to the electromagnetic vector potential: *we are rediscovering electromagnetism, from a novel point of view!* We know that from $A_t(t,\vec{x})$, $\vec{A}(t,\vec{x})$ derive field strengths $\vec{E}$ and $\vec{B}$, namely the electric and magnetic fields. The novel view offers a "geometric way" to construct them, which provides a conveniently generalizable approach if we aim at applying a similar principle of gauge invariance to a more complcated internal symmetry e.g. a non commutative (non abelian) symmetry of Isospin-type. This method consists in noting that covariant derivatives D_t and D_j do *not commute* among each other. Their commutators, applied to any field carrying the charge q, amounts to a multiplication by field strengths

$$[D_j, D_k]\,\psi(x) \;=\; -\,i\,e\,q\,[\nabla_j A_k(x) - \nabla_k A_j(x)]\,\psi(x) \equiv -\,i\,e\,q\,\epsilon_{jkl} B_l(x)\,\psi(x) \tag{3.25}$$

$$[D_t, D_k]\,\psi(x) \;=\; -\,i\,e\,q\,[\partial_t A_k(x) \,-\, \nabla_k A_t(x)]\,\psi(x) \equiv +\,i\,e\,q\,E_k(x)\,\psi(x) \tag{3.26}$$

where we recognize a magnetic field in eqn. (3.25) and an electric field in eqn. (3.26)[1], both invariant under transformations (3.24). In a non abelian framework we will not get invariant fields but "only" get only "covariant fields in a sense similar to (3.22), (3.23).

Conclusion: gauge invariance + minimal coupling ⇒ interaction, reproduces electrodynamics!

The generalization to gauge invariance for symmetries of isospin-type has successfully led to the formulation of the three relevant interactions: electromagnetic, weak, and strong among known particles at nowadays accessible energies, with a unified conceptual framework: 1) Quantum Chromodynamics as the theory of strong interaction at the quark level and 2) the Glashow-Weinberg-Salam electroweak theory describing weak and electromagnetic interactions in a combined way. The latter describes in a unified way phenomena so widely different at low energy as electromagnetism and β decay. They happen to be very different because the W (and Z) boson(s) are fairly massive whereas the photon is massless. What does trigger this diferentiation? How can one remain massless whereas the other(s) become(s) massive?

In the simplest field theories, the masses can be read on the Lagrangian by focussing on the quadratic terms in the fields (non-derivative terms). One may think: let us put a stand-alone mass term " by hand" in the Lagrangian to massify the bosons of the gauge field(s). The relativistic invariant combination would read:

$$\mathcal{L}_{mass} = \frac{1}{2}\,M^2\,\left(A_t^2 - \vec{A}^2\right) \tag{3.27}$$

Alas! as one immediately sees this is **not** invariant under gauge transformations (3.24): gauge invariance *seems to prevent* vector bosons associated to a gauge field - let's name

[1] A_t shall be identified with minus the usual electromagnetic scalar potential.

them gauge bosons - from being massive... One has to find a way which respects gauge invariance. In this persepctive let's remind also the observation made at the end of section 2. You must feel flawn at this point: you've been baffled with symmetries by this talk for quite a while and it so far lead you on that apparent no-go track... But wait! the visit of the Castle of Symmetries is not finished yet: there remains perhaps the cutest wing.

4 Spontaneous Symmetry Breaking

One speaks about Spontaneous Symmetry Breaking (SSB) when the equations governing the dynamics are left invariant under a symmetry but their solutions are not. The renown theorist Coleman used the better wording of "hidden symmetry". You may have met SSB in several occasions already, e.g. in classical mechanics of continuous media, in theory of elasticity (which is namely a classical field theory after all!). Consider a cylindric metallic rod modelled by an homogeneous isotropic medium. Apply a longitudinal strain at each extremity of the rod, along its direction: the problem has cyndric symmetry. Small to moderate strains probe the so-called linear regime - a perturbative regime: the rod responds to the strain by reducing its length yet w/o changing its shape, thereby reflecting the symmetry of the equations. Beyond some critical value hovewer, the rod find it energetically more favorable to instead buckle. Its shape no longer reflects the symmetry of the problem. Manifest symmetry is grasped only by considering all the possible orientations of the plane in which the bucking may occur.

Interestingly, in QM concerned with finitely many degrees of freedom, the phenomenon does *not occur* (if the Hamiltonian is left invariant by some symmetry, the fundamental is non degenerate and symmetric...). However in QFT concerned anew with infinitely many degrees of freedom, it can happen "again" (NB: another example in condensed matter physics is spontaneous magnetization below the Curie temperature. More generally, QFT and the theory of so-called 2^{nd} order phase transitions in condensed matter and statistical physics is a domain where both disciplines remarkably cross-fertilized). Let us focus on continuous symmetries. Instead of a quantum discussion that would be too complicated here, we stick to claasical field theory. That is suggestive enough to describe ideas.

4.1 SSB w/o gauge bosons: the Nambu-Goldstone phenomenon

Let us consider the simplest example of one complex scalar field with the following potential (of so-called "mexican hat" shape):

$$V(\phi) \;=\; \lambda\left(\phi^*\phi - \frac{1}{2}\,v^2\right)^2 \tag{4.28}$$

$\lambda > 0$ is a dimensionless coupling parameter, and $v > 0$ has dimension of a mass. The origin $\phi = 0$ is a local *maximum* of V not minimum; the minima of V are degenerate and correspond to $\phi^*\phi = v^2/2 > 0$, a circle in the complex $\{\phi\}$ plane: the global phase reparametrization symmetry of $V(\phi)$ is *spontaneously broken*. Let us choose one of these

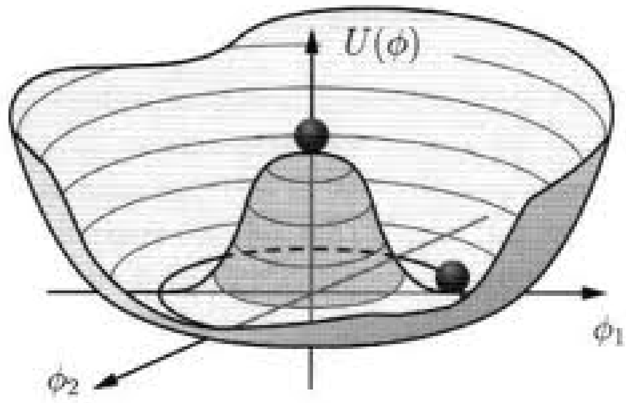

Figure 2: Mexican hat potential. ("U" on the figure corresponds to "V" in the text!)

minima, say $\phi_0 = v/\sqrt{2}$, define $\widetilde{\phi} = \phi - \phi_0 = (\widetilde{\phi}_1 + i\widetilde{\phi}_2)/\sqrt{2}$ and Taylor expand $V(\phi)$ in powers of $\widetilde{\phi}_{1,2}$. One gets:

$$V(\phi) \quad = \quad \lambda v^2\, \widetilde{\phi}_1^2 + (\textbf{no}\ \widetilde{\phi}_2^2 \text{ term}) + \cdots \tag{4.29}$$

where "$\cdots$" stands for "beyond quadratic" terms in the fields. One reads the mass spectrum on eqn. (4.29). The field $\widetilde{\phi}_1$ corresponds to a massive spinless boson in the spectrum, with mass squared $m_1^2 = 2\,\lambda v^2$. The important feature to stress here is *the absence of mass term for* $\widetilde{\phi}_2$: the massless boson associated with $\widetilde{\phi}_2$ is named a **Nambu-Goldstone (NG) mode**. The existence of a NG boson is the consequence of the spontaneous breaking of the global phase reparametrization symmetry of $V(\phi)$: the potential being symmetric and the set of minima continuously degenerated, the potential has "a (curvilinear) flat" direction along the degenerate set of minima to which thus corresponds a "vibration mode with vanishing mass according to the mechanical analogy formulated at the end of section 1.

This phenomenon is very general in field theory and can be extended to more involved situations. The global symmetry corresponds a continuous group G, but the constant field configurations of lowest energy ϕ_0 are not individualy invariant under all transformations operated by elements of G. For each given ϕ_0, only a subgroup H of G lets it invariant. Schematically, transformations g of a continuous group can be seen as finite iterations of infinitesimal transformations, i.e transformations of the form

$$g = e^{\omega} \simeq 1\!\!\mathrm{I} + \omega + \cdots$$

where $\omega = \sum_a \omega_a\, T_a$ is a infinitesimal linear combination of a set of matrices T_a named generators (think of the angular momentum in QM: for the rotation group the generators identify with the components of the angular momentum operator). Among them let us call "broken generators" a maximal set of linearly independent generators of the infinitesimal transformations of G which do not let a given ϕ_0 invariant. The nambu-Goldstone phenomenon states that **to each broken generator there corresponds a massless spinless Nambu-Goldstone boson, labelled with quantum number(s) associated**

to this broken generator. This labelling tells how the NG modes transform into each other by the symmetry transformations of the subgroup H of the symmetries of "the vacuum": this sheds light on the way such guys interact with other particles in the spectrum.

For example, in the physcis of the strong interaction there is namely a broken global symmetry of this kind named the chiral flavor symmetry. In the approximation considering only two quarks "flavors" u, d building protons and neutrons and neglecting the mass of these quarks, the pions are the corresponding Goldstone modes. The pions are not quite massless because in particular of of the non vanishing u and d quark masses, but these masses are small enough, in comparison with the typical energy scale characteristic of the binding of the quarks, to not wreck the picture completely: the latter is only deformed a little bit - as if the "circle of minima of the potential V was a little tilted away from horizontality". Extending the picture by including the s quark extends the symmetry considered and its spontaneous breaking, leading to the interpretation of the Kaons as Golsdtone modes too. The mass of the s quark being quite larger than the ones of u and d, the deformation induced by the account of s mass warps the picture somewhat more but the latter remains roughly OK.

4.2 SSB in presence of gauge bosons: the Higgs mechanism

Actually the NG phenomenon manifests itself in the way described *only in absence of gauge bosons* that would carry the quantum numbers of at least some of the broken generators. In the opposite case, there is a distinctive way out. To have an insight on what happens let us reconsider the the previous example 4.1 with its complex scalar field and its mexican hat potential, and couple it minimally to a gauge field $(A_t, \vec{A})$ as described in 3.1. We obtained something gauge invariant under local phase reparametrization, so a fortiori it is symmetric under global ones. Let us consider the "covariant kinetic" term:

$$\mathcal{L}_{kin} \;=\; (D_t\phi)^\dagger (D_t\phi) \,-\, \left(\vec{D}\phi\right)^\dagger \left(\vec{D}\phi\right) \tag{4.30}$$

where

$$\left(\vec{D}\phi\right)^\dagger \left(\vec{D}\phi\right) = \sum_{j=x,y,z} (D_j\phi)^\dagger (D_j\phi)$$

The usual relativistic invariant combination (4.30) treats time and space dependences on an equal footing. Let us label each of the space and time derivatives by a common label $\mu = t, x, y, z$. Here the symbol † stand for complex conjugation in the classical field theory case; we use it to facilitate the leap to the quantum framework as well as to generalizations to more complcated symmetries. Let us expand the term

$$\begin{aligned}&(D_\mu\phi)^\dagger \Big((\partial_\mu - i\,e\,q\,A_\mu)\,\phi \Big) \\ &= \;(\partial_\mu\phi)^\dagger (\partial_\mu\phi) + \,i\,e\,\left[(\partial_\mu\phi)^\dagger q\,\phi \,-\, \phi^\dagger q\,(\partial_\mu\phi) \right] A_\mu + e^2 \,\left(\phi^\dagger q^2\,\phi\right)\,(A_\mu A_\mu)\end{aligned} \tag{4.31}$$

The first term in the r.h.s. reproduces the kinetic terms of a free complex field similar to the first term of eqn. (1.11) seen at the end of section 1. The 2^{nd} term involves a trilinear coupling. The term on which our attention shall focus here is the last term. In it, let us put ϕ equal to a constant value ϕ_0 which minimizes the mexican hat potential, thereby spontaneously breaking the phase reparametrization symmetry. The term obtained reads:

$$\mathcal{L}_m \;=\; e^2 \left(\phi_0^\dagger\, q^2\, \phi_0\right)\, (A_\mu A_\mu) \;=\; \frac{1}{2}\, e^2\, q^2\, v^2\, (A_\mu A_\mu) \tag{4.32}$$

which looks the same as (3.27) i.e. the gauge field is given a mass

$$M \;=\; e\, |q|\, v$$

but this time it is done in a gauge invariant way, since this term emerges from the gauge invariant setup (4.30)!

1. What does make it gauge invariant?
   This is namely the accompanying interaction terms between A_μ and ϕ or, if one prefers, $\widetilde{\phi} = (\phi - \phi_0)$ in eqn. (4.31).

2. What about the counting of degees of freedon stressed at the end of 2.2.1 in point 2?
   The degree of freedom described by $\widetilde{\phi}_2$ becomes the 3^{rd} polarization state of the vector field A_μ, whereas the particle spectrum no longer contains the massless spin NG mode: in a sense the NG mode has been eaten by the A_μ which has become massive and has acquired a 3^{rd} polarization state!

The above two issues happen to be related to each other. The phenomenon is best captured if one reparametrizes the scalar field ϕ in "polar field coordinates" as

$$\phi(t, \vec{x}) = \left(\phi_0 + \frac{\rho(t, \vec{x})}{\sqrt{2}}\right)\, e^{i\,\theta(t,\vec{x})} \tag{4.33}$$

One notices that the phase term in eqn. (4.33) looks formally like a gauge transformation (3.18). Having in mind this mechanical analogy with oscillation modes of small amplitudes seen in section 1, let us Taylor expand as

$$\widetilde{\phi}_1 \;=\; \rho\, \cos\theta \;\sim\; \rho + \cdots \tag{4.34}$$

$$\widetilde{\phi}_2 \;=\; (v + \rho)\, \sin\theta \;\sim\; v\,\theta + \cdots \tag{4.35}$$

One sees, loosely speaking, that the field $\widetilde{\phi}_2$ which described the NG in the previous case 4.1, formally corresponds to a gauge transformation. Performing the inverse gauge transformation defined by

$$f(x) \;=\; e^{-\,i\,\theta(x)} \;=\; e^{-i\,\widetilde{\phi}_2(x)/e\,q\,v} \tag{4.36}$$

according to eqn. (3.18), the degree of freedom carried by $\widetilde{\phi}_2$ is removed away from the "sector" of the scalar fields. This transformation moves the degree of freedom carried by $\widetilde{\phi}_2$ "inside the gauge tranformed field A'" according to eqn.(3.24)

$$A_\mu \;\longrightarrow\; A'_\mu = A_\mu - \frac{\partial_\mu \widetilde{\phi}_2}{e\,q\,v}$$

Concomitantly the above gauge transformation (3.18),(4.36) removes the extra terms

$$\frac{1}{2}\left(\partial_\mu\widetilde{\phi}_2\right)\left(\partial_\mu\widetilde{\phi}_2\right) + e\,q\,v\,(\partial_\mu\widetilde{\phi}_2)\,A_\mu$$

from the first and second terms in $\mathcal{L}_{kin}$ away from the scalar kinetic term, and moves them to combine them with the A mass term (4.32), so as to build the mass term for the gauge-transformed field A' which becomes

$$\frac{1}{2}\,e^2q^2v^2A_\mu A_\mu \;\longrightarrow\; \frac{1}{2}\,e^2q^2v^2\left(A_\mu - \frac{\partial_\mu\widetilde{\phi}_2}{e\,q\,v}\right)\left(A_\mu - \frac{\partial_\mu\widetilde{\phi}_2}{e\,q\,v}\right) = e^2q^2v^2\,A'_\mu\,A'_\mu$$

i.e. has the same form in both gauges: it is indeed gauge invariant! Last, there is one massive real scalar field described by $\sim\widetilde{\phi}_1$ with mass $\sqrt{2\lambda}\,v$, coupled to the massive vector field A'_μ, according to the remaining unexplicited terms of eqn. (4.31). Furthermore, the way the longitudinal polarization state of A'_μ couples to the other player in the game keeps the memory that "it was a NG mode in another life".

Conclusion: **a dynamical rearrangement of degrees of freedom has happened, whereby two massless helicity 1 degrees of freedom + one massless spinless NG mode have combined together to build a massive spin 1 boson: this is mechanism is the Higgs mechanism!** - more fairly coined Anderson, Brout-Englert, Higgs, Guralnik-Kibble-Hagen mechanism. In the simplest version of the mechanism there is (at least) one accompanying spinless massive boson present in the spectrum, corresponding to $\widetilde{\phi}_1$ in the above model: the so-called Higgs boson. The 125 GeV resonance found by the Atlas and CMS experiment a year ago looks very much like such a guy.

Let us stress in addition that:

- the statement does not claim that all gauge bosons shall become massive.
  There may be gauge fields carrying the quantum numbers of unbroken generators (in the terminology of the previous subsection), they are then associated with no Goldstone modes: those gauge bosons are left massless. This is namely what happens to the photon field in the Standard Model.

- alternatively the statement does not claim that all Goldstone modes shall be eaten. If there are many Goldstone bosons and if no gauge boson carries the same quantum numbers as some of them, these Goldstone modes will remain in the particle spectrum as spinless bosons. Whether this hypothetical possibility is met in nature is so far an open question.

4.2.1 What about fermion masses? The fermion mass and mixing puzzle

The electroweak gauge symmetry in the Standard Model happens to forbids fermions as well to be given a mass term in a naive way by a stand-alone mass term in the Lagrangian: this has to be provided within a dynamical gauge invariant framework.

The scalar field involved in the above described Higgs et al. mechanism *can* also play a key role in the fermion mass and mixing issue, although this concomitant phenomenon shall be conceptually *disantangled* from the Higgs et al. mechanism per se - strickly speaking the latter minds the sole vector bosons.

As the problem is formulated in the Standard Mdel (SM) and some similar extensions, the Higgs field actually does the job for fermions “to some extend”. This damper means that the SM description *accomodates* the fermion masses and mixings etc. in a way consistent with electroweak gauge invariance. Mixing refers to the following phenomenon. On one hand the mass eigenbasis of fermion fields is the basis in which the fermion mass terms in the Lagrangian look simple (diagonal mass matrices). Yet it is not the basis in which the minimal coupling to the weak boson W is formulated in simple terms. The two bases are not aligned onto each other. The mismatch between the two is what is called fermion mixing; it leads to interesting observable effects, both for quarks and which are presently studied in LHCb, and for neutrinos for which it leads to the phenomenon of neutrino oscillations. The massification of fermions together with mixing are parametrized in the SM in a way consistant with gauge invariance, but there are as many parameters to adjust to experiment as there are masses and mixing angles. This let theorists unsatisfied. Alternatively there may be some yet-to-discover piece of dynamics at work, carrying specifically about this issue and providing a deeper explanation of why things look the way they look. This might however be hard to probe in a way independent of the study of the fermion mass and mixing pattern, if distinctive signals of this new dynamics are not expected within a reachable energy range. All this is another story, one of the hot and challenging puzzles remaining to be solved. It would be worth a lecture on its own.

For an accessible textbook introducing to gauge theories, SSB, the Higgs et al. mechanism, and how this fits into the SM, see for example the nice ref. [1].

5 From Lagrangian to observables

Now comes an important question: how does this sophisticated field theoretical picture make contact with experimental physics?

QFT was elaborated with the concern of providing, not a mere aesthetic abstraction, but a predicitive setup to effectively - and efficiently! - compute observables that can in parallel be measured in high energy experiments: in particular transition probability rates for particle processes. Going from the Lagrangian, which encodes the modelling of the quantum fields and their interactions, to numerical calculations of transition probability rates is a technicaly laborious task. A qualitative description to give a foretaste skipping as many technicalities as possible is challenging, both to tell and to listen to or read... This brief fourth section is an attempt in this direction. It aims at teasing your curiosity more than really explaining things, by conjugating fierce handwavings and a few weirdo formulas... If you get lost keep walking straight the exit is in front of you!

5.1 The interaction picture

In the theory accounting for interactions, the interacting fields ϕ obey Heisenberg's equation $i\,\partial_t \phi = [H, \phi]$ where H is the complete Hamiltonian. The effects of interactions are encoded implicitely in the interacting field ϕ in an intricate way next-to-unexploitable in practice. Can one unravel this intricated content explicitely in a way that can be conveniently treated in perturbation theory around free fields? The free fields are chosen to obey the non interacting equation $i\,\partial_t \phi_f = [H_0, \phi_f]$ where H_0 is the free Hamiltonian so as to describe free particles such as prepared independently of each other in a remote past and sent toward each other to undergo a collision. The idea is to build a unitary mapping U between free fields and interacting field of the form

$$\phi(t, \vec{x}) = U(t)\, \phi_f(t, \vec{x})\, U(t)^\dagger \tag{5.37}$$

One shows that $U(t)$ fulfils an eqn. of the form

$$i\ (\partial_t U(t))\ U^\dagger(t) = H_{int}(\{\phi_f\})$$

where the interaction term $H_{int} = H - H_0$ is expressed as space integrals of polynomials in the free fields and their derivatives. Very rouglhy $U(t)$ takes a sort of exponential form

$$U(t) \sim \exp\left\{-i \int_0^t dt' H_{int}(\{\phi_f\})\right\}$$

(the correct result happens to be not quite an ordinary exponential, but this is moraly good enough for our qualitative level of discussion). This is the so-called interaction picture. Taking remote time limits $t \pm \infty$ allows to relate the interaction fields to "in" free fields decribing incoming particles in a remote past on one hand, and between the same interaction fields and "out" free field describing the outgoing particles to be dectected in a remote future on the other hand. "Remote" here is relative to the time duration during which interation takes place. One thereby builds a mapping of the basis of incoming states in a process onto the one of outgoing states from this process, involving a unitary operator named the S-matrix (S for scattering) matrix "morally given by"

$$S\ \sim\ \exp\left\{-i \int_{-\infty}^{+\infty} dt' H_I(\{\phi_{in}\})\right\}\ \sim\ \exp\left\{-i \int d^4x \mathcal{L}_{Int}(\{\phi_{in}\})\right\} \tag{5.38}$$

where $\mathcal{L}_{int}$ stands for the interaction terms in the Lagrangian. To the process

$$\text{particle}_1(p_1) + \text{particle}_2(p_2) \to \text{particle}'_1(p'_1) + \cdots + \text{antiparticle}'_n(p'_n)$$

where the p_i, p'_j stand for the four-momenta of the aprticles, corresponds the transition amplitude

$$<\ \text{particle}'_1(p'_1), \cdots, \text{antiparticle}'_n(p'_n)\ out\,|\,\text{particle}_1(p_1),\ \text{particle}_2(p_2)\ in\ >$$

given by the S-matrix element

$$<\ \text{particle}'_1(p'_1), \cdots, \text{antiparticle}'_n(p'_n),\ in\,|S|\,\text{particle}_1(p_1), \text{particle}_2(p_2),\ in\ >$$

5.2 From Lagrangian to transition amplitudes

The states $|\{\text{particle}'\ j\}_{j=1,\cdot,n}\},\ in\ >$ can in turn be expressed in terms of creators acting on the vacuum:

$$|\,\text{particle}'_1(p'_1),\cdots,\text{antiparticle}'_n(p'_n)\ in\ >= b'^{\dagger}_1(p'_1)\cdots d'^{\dagger}_n(p'_n)|\,0\ >$$

the creators and annihilators can be expressed as inverse Fourier transforms of the free "in fields" ϕ and $\phi^\dagger$ by inverting eqn. (1.7). Everything thus boils down to the evaluation of the vacuum expection value of the Fourier transform (w.r.t. the four-momenta of the incoming and outgoing particles) of some multiplicative string of "in" fields as the quantity to be computed looks like

$$\int d^4x_1\, d^4x_2 \prod_{j=1}^{n} d^4x'_j\, e^{-\,i\,(p_1.x_1+p_2.x_2-\sum_{j=1}^{n} p'_j.x'_j)}$$
$$<\ 0\,|\phi_{in}(x_1)\phi_{in}(x_2)\phi^\dagger_{in}(x'_1)\cdots\phi^\dagger_{in}(x'_n)e^{i\int\mathcal{L}_{int}(\{\phi_{in}\})}|\,0\ > \qquad (5.39)$$

The exponential $e^{i\int\mathcal{L}_{int}(\{\phi_{in}\})}$ in expression (5.39) is then Taylor expanded - which is named perturbative expansion - at the order corresponding to the accuracy aimed for the calculation. Expression (5.39) is calculated explicitely as a numerical (in general complex) function of the four-momenta, spins, charges etc. of the particles involved. The transition amplitude is directly proportional to this quantity in a simple manner.

5.3 From transition amplitudes to cross sections

The transition probability per time unit and space volume unit $dw_{i\to f}/(dt\, d^3x)$ is equal to the modulus squared of the transition amplitude obtained from eqn. (5.39). It has dimension $[Length]^{-3}\,[Time]^{-1}$. It is a kind of generalization of Fermi Golden's rule learnt (or to be soon learnt) in QM class on time-dependent perturbation theory. In order to compare theoretical calculations and experimental particle physics measurements, it is convenient to present transition probabilities of collision processes in the form of *cross sections*. A cross section is defined as a transition probability per time unit, per target particle and per unit of incident projectile flux. The projectile flux counts the number of projectiles particles passing per surface unit and per time unit, it has dimension $[Length]^{-2}\,[Time]^{-1}$. Thus a cross-section has dimension

$$\frac{[Time]^{-1}}{[Length]^{-2}\,[Time]^{-1}} = [Length]^2$$

i.e. a surface. Following Berkeley's Physics Course on QM, an intuitive classical picture of what a cross section represents could be visualized as follows. Imagine a disk whose area is equal to this cross section, centered on a target particle, and oriented perpendicular to the incident direction of projectiles. Imagine a cylinder whose cross section is this disk and whose axis is the incident direction of projectiles. The target particle will "hit" any projectile coming accross the target particle within this cylinder. In practice the values of cross sections in high energy physics are minuscules. The reference unit for cross sections is the *barn*, $1\ b = 10^{-28}$ m^2. Actually this order of magnitude is roughly the one suitable in nuclear physics. High energy physics now explores a regime where the interesting phenomena are extremely rare and for which the cross sections are counted in *femto-barns* or less, where the prefix *femto* $\equiv 10^{-15}$.

The cross section is obtained from the probability rate $dw_{i\to f}/(dt\, d^3x)$ by dividing the latter with a kinematical, relativistic invariant factor involving the four-momenta p_1 of the target and p_2 of the projectile, named flux factor, whose relativistic invariant form $[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}$ is best suited when both incomers move towards each other as in LHC collisions seen in the lab frame, and which reads $m_1\|\vec{p}_2\|$ in the rest frame of the target particle. More precisely one obtains the fully differential cross section w.r.t. all the kinematical variables of all the final state particles. One integrates over the variables not measured or projected out to provide all sorts of distributions which can be "directly" compared with the corresponding histogramming outputs of experimental data. Here is where theory and experiments meet, at last!

There is nothing more that can be seen without going down into the stables of the castle and putting one's feet and hands in the manure, so here ends the visit of the Castle of Particle Fields and Symmetries. We hope you have enjoyed it. Please don't forget the guide!

Epilogue, acknowledgements & embryonic bibliography

This lecture aimed at providing a very partial overview of concepts and ideas at work in theoretical particle physics. It cannot stand in for a serious teaching of the matter in any respect. May it spark off or feed your wish to learn more about it! I heartily thank my colleagues and friends F. Thuillier and J.-P. Guillet from LAPTh who helped me select the material of this lecture and frame it in a format hopefully accessible for L3 and M1 students.

There are of course *many* valuable textbooks on QFT, likewise on SM building. Only one reference is quoted herebelow, reader-friendly and accompanied by a sister book of exercises with solutions, to not overload you wih too many ones!

References

[1] T.P. Cheng & L.F. Li, *Gauge Theory of Elementary Particle Physics*, ed. Oxford.