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1. The shower main characteristic can be obtained using a simple Heitler model. Consider a shower initiated by a photon - a particle having only electromagnetic interaction. Let E_0 be the energy of the primary particle and consider that the electrons, positrons and photons in the cascade always interact after travelling a certain atmospheric depth d, and that the energy is always equally shared between the two particles. With this assumptions, we can schematically represent the cascade as in figure 1(a).

a) Write the analytical expressions for the number of particles and for the energy of each particle at depth X as a function of d, n and E_0.

b) The multiplication of the cascade stop when the particles reach a critical energy, E_c (when the decay probability surpasses the interaction probability). Using the expressions obtained in the previous exercise, write as a function of E_0, E_c and $\lambda = d/\ln(2)$, the expressions, at the shower maximum, for:

   i) the average energy of the particles;
   ii) the number of particles, N_{max};
   iii) the atmospheric depth X_{max}.

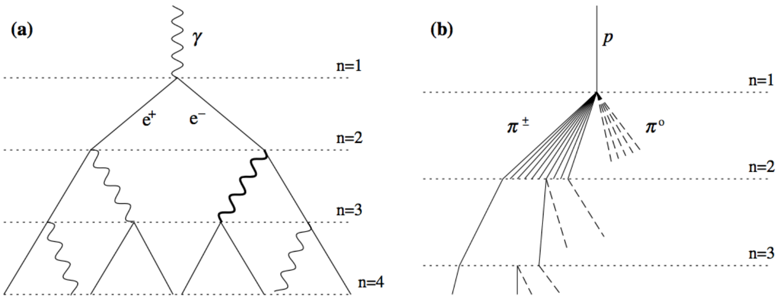

Figure 1: Schematic views of (a) an electromagnetic cascade and (b) a hadronic shower. In the hadron shower, dashed lines indicate neutral pions which do not re-interact, but quickly decay, yielding electromagnetic subshowers (not shown). Not all pion lines are show n after the n = 2 level. Neither diagram is to scale.

2. Consider a shower initiated by a proton of energy E_0. We will describe it with the simple model of figure 1(b): after each depth d an equal number of pions, n_π, and each of the 3 types is produced: π^0, π^+, π^-. Neutral pions decay through $\pi^0 \to \gamma\gamma$ and their energy is transferred to the electromagnetic cascade. Only the charged pions will feed the hadronic cascade. We consider that the cascade ends when these particles decay as they reach a given decay energy E_{dec}, after n interactions, originating a muon (plus an undetected neutrino).

a) How many generations are needed to have more that 90% of the primary energy, E_0 in the electromagnetic component?

b) Assuming the validity of the superposition principle, according to which a nucleus of mass number A and energy E_0 behaves like A nucleons of energy E_0/A. Derive expressions for:

i) the depth where this maximum is reached, X_{max};

ii) the number of particles at the shower maximum;

iii) the number of muons produced in the shower, N_μ.

3. The transparency of the Universe to a given particle depends critically on its nature and energy. In fact, whenever it is possible to open an inelastic channel of the interaction between the *travelling* particle and the Cosmic Microwave Background, its mean free path diminishes drastically. Assuming that the only relevant phenomena that rules the mean free path of the *travelling* particle is the CMB (CνB), estimate the order of magnitude of the energies at which the transparency of the Universe changes significantly, for:

a) Photons;

b) Protons;

c) Neutrinos.

$$\langle E_{\gamma_{CMB}} \rangle \approx 1.2 \text{ meV} \; ; \; \langle E_{\nu_{C\nu B}} \rangle \approx 0.56 \text{ meV}$$
$$m_e = 511 \text{ KeV} \; ; \; m_{\Delta^+} \approx 1.2 \text{ GeV} \; ; \; m_p \approx 1 \text{ GeV} \; ; \; m_\pi \approx 0.14 \text{ GeV} \; ; \; m_{Z^0} \approx 91 \text{ GeV}$$

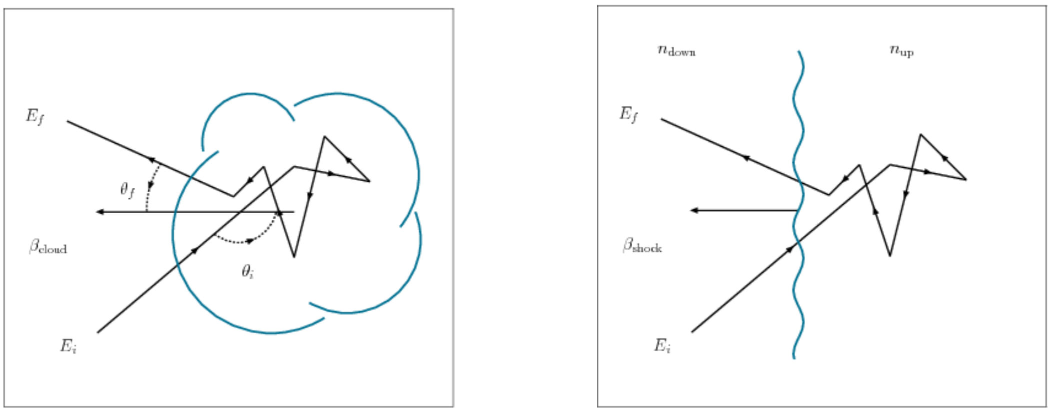

Figure 2: Sketch of a collision of a charged particle with a moving: (left) magnetic cloud - second order fermi acceleration mechanism; (right) shock wave - first order fermi acceleration mechanism.

4. Calculate, for the first and second order Fermi acceleration mechanism, how many times has the particle to cross the cloud (shock) to gain a factor 10 on its initial energy.

a) Assume realistic values of $\beta = 10^{-4}$ for the magnetic cloud and $\beta = 10^{-2}$ for the shock wave.

b) Repeat the previous exercise assuming $\beta = 10^{-4}$ for both acceleration mechanisms.

5. In the *isothermal* approximation, the depth x of the atmosphere at a height h (i.e., the amount of atmosphere above h) can be approximated as

$$x \simeq X e^{-h/7\,\text{km}},$$

with $X \simeq 1030$ g/cm^2. Consider a shower is generated by a gamma ray of E= 1 TeV penetrating the atmosphere vertically. Taking the radiation length X_0 of air is approximately 36.6 g/cm^2 (440 m) and the critical energy E_c to be about 88 MeV:

a) Calculate the height h_M of the maximum of the shower in the Heitler model and in the Rossi approximation B (see table 1).

b) Assume that 2000 useful Cherenkov photons per radiation length are emitted by charged particles in the visible and near UV. Compute the total number N_γ of Cherenkov photons generated by the shower. (note: the critical energy is larger than the Cherenkov threshold).

	Incident electron	Incident photon
Peak of shower t_{max}	$1.0 \times (\ln y - 1)$	$1.0 \times (\ln y - 0.5)$
Centre of gravity t_{med}	$t_{max} + 1.4$	$t_{max} + 1.7$
Number of e^+ and e^- at peak	$0.3y/\sqrt{\ln y - 0.37}$	$0.3y/\sqrt{\ln y - 0.31}$
Total track length	y	y

Table 1: Shower parameters according to Rossi approximation B. $y = E/E_c$; unit of length is the radiation length.

c) Compute how many photons per square meter arrive to a detector at a height h_d of 2000 m, supposing that the average attenuation length of photons in air is 3 km, and that the light pool can be derived by a opening of $\sim 1.3°$ from the shower maximum (1.3° is the Cherenkov angle and 0.5° comes from the intrinsic shower spread). Consider that the Cherenkov photons are all emitted at the centre of gravity of the shower (use the Rossi approximation B).

d) Cherenkov telescopes have usually collection areas of $\sim 100\,\text{m}^2$. Compute the number of Cherenkov photons collected by a telescope considering an average reflectivity of the mirrors (including absorption in transmission) of 70%, and a photodetection efficiency (including all the chains of acquisition) of 20%.

e) Redo the calculations for $E = 50$ GeV, and comment.

6. The Cosmic Microwave Background fills the Universe with photons with a peak energy of 0.37 meV and a density of $\rho \sim 400/\text{cm}^3$. Determine the interaction length of such protons in the Universe considering a mean cross-section above the threshold of 0.6 mb.

7. Pions and muons are produced in the high atmosphere, at a height of some 10 km above sea level, as a result of hadronic interactions from the collisions of cosmic rays with atmospheric nuclei. Compute the energy at which charged pions and muons respectively must be produced to reach in average the Earth's surface.

$$\begin{aligned} m_\pi &= 139.57\,\text{MeV}/c^2 \quad ; \quad c\,\tau_\pi = 7.8\,\text{m} \\ m_\mu &= 105.66\,\text{MeV}/c^2 \quad ; \quad c\,\tau_\mu = 658.6\,\text{m} \end{aligned}$$

8. A neutral pion decays almost exclusively into two photons. Assuming that the π^0 has an energy E_π what is, in the LAB framework:

a) The minimum and maximum energy that the photons can have as a function of the π^0 energy.

b) What would be the energy spectrum of the produced photons.

Solutions

1.

a)

$$\begin{aligned} X &= n \times d \\ N &= 2^n \\ E_i &= \frac{E_0}{2^n} \end{aligned}$$

b) i)

$$E = E_c$$

ii)

$$N_{max} = \frac{E_0}{E_c}$$

iii)

$$X_{max} = n_{max} \times d = \frac{\ln(N_{max})}{\ln(2)} d = \lambda \ln\left(\frac{E_0}{E_c}\right)$$

2.

a)

$$n = \frac{\log(0.1)}{\log(2/3)} \approx 5.7 \text{ generations.}$$

b) i)

$$\begin{aligned} \text{Proton:} \quad X_{max} &= d\frac{\log\left(E_0/E_{dec}\right)}{\log(n_\pi)} \\ \text{Iron:} \quad X_{max} &= \frac{d}{\log(n_\pi)}\left(\log\left(\frac{E_0}{E_{dec}}\right) - \log(A)\right) \end{aligned}$$

ii)

$$\begin{aligned} \text{Proton:} &\quad N_{max} = n_\pi^{n_{dec}} = \frac{E_0}{E_{dec}} \\ \text{Iron:} &\quad \text{same result as proton} \end{aligned}$$

iii)

$$\begin{aligned} \text{Proton:} &\quad N_\mu = \left(\frac{E_0}{E_{dec}}\right)^\beta \\ \text{with} &\quad \beta = \frac{\log\left(\frac{2}{3}n_\pi\right)}{\log(n_\pi)} \\ \text{Iron:} &\quad N_\mu = A^{1-\beta}\left(\frac{E_0}{E_{dec}}\right)^\beta \end{aligned}$$

3.

a)

$$E_\gamma = \frac{m_e^2}{E_b} \sim 10^{14}\,\text{eV}$$

b)

$$E_p = \frac{m_\pi^2 + 2m_p m_\pi}{4E_b} \sim 6 \times 10^{19}\,\text{eV}$$

c)

$$E_\nu = \frac{m_{Z^0}^2}{4E_b} \sim 10^{24}\,\mathrm{eV}$$

4.

a)

$$\begin{aligned} n(\varepsilon \propto \beta; \beta = 10^{-2}) &\approx 2.3 \times 10^2 \text{ cycles} \\ n(\varepsilon \propto \beta^2; \beta = 10^{-4}) &\approx 2.3 \times 10^8 \text{ cycles} \end{aligned}$$

b)

$$\begin{aligned} n(\varepsilon \propto \beta; \beta = 10^{-4}) &\approx 2.3 \times 10^4 \text{ cycles} \\ n(\varepsilon \propto \beta^2; \beta = 10^{-4}) &\approx 2.3 \times 10^8 \text{ cycles} \end{aligned}$$

5.

a)

$$\begin{aligned} \text{Heitler:} \quad & h_M \approx 4.9\,\mathrm{km} \\ \text{Rossi:} \quad & h_M \approx 8.6\,\mathrm{km} \\ \text{Heitler:} \quad & h_M \approx 9.5\,\mathrm{km} \end{aligned}$$

b)

$$N_\gamma^{total} = \left(\frac{E}{E_c}\right) N_\gamma^{Ch} \approx 2.27 \times 10^7 \text{ photons}$$

c)

$$n_\gamma = \frac{N_\gamma|_{ground}}{A} = \frac{N_\gamma\, e^{-(h_{med}-h_d)/3}}{\pi\, r_p^2} = 116 \text{ photons m}^{-2}$$

d)

$$n_\gamma^{det} = n_\gamma\, A_{det}\, \varepsilon_{ref}\, \varepsilon_{acq} \approx 1629 \text{ photons}$$

e)

Energy [GeV]	t_{med}	X_{med} [g cm^{-2}]	h_{med} [km]	r_p [m]	N_γ [ph]	n_γ [ph m^{-2}]	n_γ^{det} [ph]
50	7.5	276.0	9.22	163.7	1.1×10^6	1	17
1000	10.5	385.7	6.88	110.6	2.2×10^7	116	1629

6.

$$L_{int} \simeq 4\,10^{24} \text{ cm } \simeq 1.3 \text{ Mpc}$$

7.

$$\begin{aligned} E_\pi &\simeq 180\,\mathrm{GeV} \\ E_\mu &\simeq 1.6\,\mathrm{GeV} \end{aligned}$$

8.

a)

$$\begin{aligned} E_\gamma^{min} &= \frac{E_\pi}{2} - \frac{P_\pi}{2} \\ E_\gamma^{max} &= \frac{E_\pi}{2} + \frac{P_\pi}{2} \end{aligned}$$

b) Uniform distribution for a fixed E_π.