Fluid Equations

$$\frac{d\rho}{dt} + \rho \vec{\nabla}_r \cdot \vec{u} = 0 \quad \text{(continuity equation)}$$

$$\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla}_r P}{\rho} - \vec{\nabla}_r \Phi \quad \text{(Euler equation)}$$

$$\vec{\nabla}_r^2 \Phi = 4\pi G \rho \quad \text{(Poisson equation)}$$

$\vec{\nabla}_r$ ← derivative wrt proper coordinate

$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_{\vec{r}} + \vec{v} \cdot \vec{\nabla}_r$ ← convective time derivative (as a quantity moves with the fluid)

↑ time derivative at fixed $\vec{r}$

$(\rho, \vec{v}, \Phi, P)$

↳ need equation for this guy

Expanding universe:

$\vec{r} = a(t)\,\vec{x}$ ← comoving coordinate

↳ scale factor ⟹ given by Friedmann equations

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

proper time

$$\vec{v}(x,t) = \frac{d\vec{r}}{dt} = \frac{da}{dt}\vec{x} + a(t)\dot{\vec{x}}$$

$$= aH\vec{x} + a\dot{\vec{x}}$$

$$= H\vec{r} + a\dot{\vec{x}} = H\vec{r} + \vec{u}(\vec{x},t)$$

↑ perturbations (peculiar velocities)

$$\boxed{\vec{v}(x,\tau) = \mathcal{H}\vec{x} + \vec{u}(\vec{x},\tau)}$$

conformal time

$\Rightarrow d\tau = \frac{dt}{a(t)}$

MMS:

$$ds^2 = -c^2dt^2 + a^2 d\vec{x}^2$$

$$= a^2\left[-c^2 d\tau^2 + d\vec{x}^2\right]$$

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\tau}$$

$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau} = aH$ conformal Hubble constant

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla}_x \quad \left(\frac{\partial}{\partial \vec{r}} = \frac{1}{a}\frac{\partial}{\partial \vec{x}}\right)$$

$$\left.\frac{\partial}{\partial r_i}\right|_t = \left.\frac{\partial x_j}{\partial r_i}\right|_t \frac{\partial}{\partial x_j}$$

$$= \frac{1}{a}\delta_{ij}\frac{\partial}{\partial x_j}$$

$\left(\frac{\partial}{\partial t}\right)_{\vec{r}}$

$$\left(\frac{\partial}{\partial t}\right)_{\vec{r}} = \left(\frac{\partial t}{\partial t}\right)_{\vec{r}}\left(\frac{\partial}{\partial t}\right)_{\vec{x}} + \left(\frac{\partial \vec{x}}{\partial t}\right)_{\vec{r}} \cdot \vec{\nabla}_x$$

$\vec{\nabla}_x = \frac{\partial}{\partial \vec{x}}$ $\quad$ $t = t(\tau, \vec{x})$

$$\left(\frac{\partial}{\partial t}\right)_{\vec{r}} = \left(\frac{\partial \tau}{\partial t}\right)_{\vec{r}}\left(\frac{\partial}{\partial \tau}\right)_{\vec{x}} + \left(\frac{\partial \vec{x}}{\partial t}\right)_{\vec{r}} \cdot \vec{\nabla}_x$$

$$= \frac{1}{a}\frac{\partial}{\partial \tau} + \left.\frac{\partial(a^{-1}\vec{r})}{\partial t}\right|_{\vec{r}} \cdot \vec{\nabla}_x$$

2

$$\left(\frac{\partial}{\partial t}\right)_{\vec{r}} = \frac{1}{a}\frac{\partial}{\partial \tau} - H\vec{x}\cdot\vec{\nabla}_x = \frac{1}{a}\frac{\partial}{\partial \tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}$$

$$\rho(\vec{x},\tau) = \bar{\rho}(\tau)\left[1+\delta(\vec{x},\tau)\right] \qquad \left(\vec{v}(\vec{x},t) = \mathcal{H}\vec{x} + u(\vec{x},\tau)\right)$$

plug all this in continuity eq.:

$$\frac{d\rho}{dt} + \rho\vec{\nabla}_r\cdot\vec{v} = 0$$

$$\left(\cancel{\frac{1}{a}\frac{\partial\rho}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\rho + \rho\frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0} \; ?\right)$$

$$\frac{\partial\rho}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}\rho + \rho\vec{\nabla}_{\vec{r}}\cdot\vec{v} = 0 \Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}_{\vec{r}}\cdot(\rho\vec{v}) = 0}$$

$$\frac{1}{a}\frac{\partial\rho}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\left[\frac{1}{a}\frac{\partial}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right]\left[\bar{\rho}(1+\delta)\right] + \frac{1}{a}\vec{\nabla}\cdot\left[\bar{\rho}(1+\delta)(\mathcal{H}\vec{x}+\vec{u})\right] = 0$$

0-th order: $(\delta, \vec{u} = 0)$

$$\left[\frac{1}{a}\frac{\partial}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right]\bar{\rho}(\tau) + \frac{1}{a}\vec{\nabla}\cdot\left[\bar{\rho}\mathcal{H}\vec{x}\right] = 0$$

$$\frac{1}{a}\frac{\partial\bar{\rho}}{\partial\tau} + \frac{1}{a}\mathcal{H}\bar{\rho}\underbrace{\vec{\nabla}\cdot\vec{x}}_{=3} = 0$$

$$\frac{\partial\bar{\rho}}{\partial\tau} + 3\mathcal{H}\bar{\rho} = 0 \qquad \text{CE for Background (mass) density}$$

$$(\bar{\rho} \propto 1/a^3)$$

1-st order: $(\delta, \vec{u} = 0)$

Can keep $(1+\delta)\vec{u}$ to get full NL level

$$\left[\frac{1}{a}\frac{\partial}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right]\left(\cancel{\bar{\rho}} + \bar{\rho}\,\delta(\vec{x},\tau)\right) +$$

BG level

$$+ \frac{1}{a}\vec{\nabla}\cdot\left[\cancel{\bar{\rho}\mathcal{H}\vec{x}} + \bar{\rho}\mathcal{H}\vec{x}\,\delta(\vec{x},\tau) + \bar{\rho}\,\vec{u}(\vec{x},\tau)\right] = 0$$

$$\delta\frac{1}{a}\frac{\partial\bar{\rho}}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\underbrace{\vec{\nabla}\bar{\rho}}_{=0} + \frac{\bar{\rho}}{a}\frac{\partial\delta}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\bar{\rho}\cdot\vec{\nabla}\delta + \frac{\bar{\rho}}{a}\left(\mathcal{H}\delta\underbrace{\vec{\nabla}\cdot\vec{x}}_{3} + \mathcal{H}\vec{x}\cdot\vec{\nabla}\delta\right)$$

$$+ \frac{\bar{\rho}}{a}\vec{\nabla}\cdot\vec{u} = 0$$

$$\underbrace{\delta\left[\frac{1}{a}\frac{\partial\bar{\rho}}{\partial\tau} + \frac{\bar{\rho}}{a}3\mathcal{H}\right]}_{=0 \text{ BG}} + \frac{\bar{\rho}}{a}\frac{\partial\delta}{\partial\tau} - \cancel{\frac{\mathcal{H}}{a}\vec{x}\bar{\rho}\cdot\vec{\nabla}\delta} + \cancel{\frac{\mathcal{H}}{a}\bar{\rho}\vec{x}\cdot\vec{\nabla}\delta} + \frac{\bar{\rho}}{a}\vec{\nabla}\cdot\vec{u} = 0$$

$$\frac{\bar{\rho}}{a}\frac{\partial\delta}{\partial\tau} + \frac{\bar{\rho}}{a}\vec{\nabla}\cdot\vec{u} = 0$$

$$\boxed{\frac{\partial\delta}{\partial\tau} + \vec{\nabla}\cdot\vec{u} = 0}$$

~~Euler eq. (EE)~~ Poisson eq. (PE) ③

$$\cancel{\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla}_r p}{\rho} - \vec{\nabla}_r \Phi}$$

$$\vec{\nabla}_r^2 \Phi = 4\pi G \rho$$

perturbation: $\Phi = \bar{\Phi} + \phi$

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla} \qquad \vec{\nabla}_r^2 = \vec{\nabla}_r \cdot \vec{\nabla}_r = \frac{1}{a^2}\vec{\nabla}^2$$

$$\frac{1}{a^2}\vec{\nabla}^2\bar{\Phi} + \frac{1}{a^2}\vec{\nabla}^2\phi = 4\pi G \bar{\rho}(1+\delta)$$

0-th order:

$$\frac{1}{a^2}\vec{\nabla}^2\bar{\Phi} = 4\pi G\bar{\rho}$$

suspect: $\bar{\Phi} \propto |\vec{x}|^2$

try: $\bar{\Phi} = C \cdot x^2$

$$\Rightarrow \frac{1}{a^2} C \cdot 6 = 4\pi G \bar{\rho}$$

$$C = \frac{4\pi G}{6} a^2 \bar{\rho} = \frac{2\pi G}{3} a^2 \bar{\rho}$$

what is?

$$\vec{\nabla} x^2 = ? = 2\vec{x}$$

$$\frac{\partial}{\partial x_i}|\vec{x}|^2 = \frac{\partial}{\partial x_i}(x^2+y^2+z^2)$$

$$= \frac{\partial}{\partial x_i}\sum_j x_j^2$$

$$= \sum_j 2 \cdot x_i$$

$$\vec{\nabla}^2(x^2) = 6$$

$$\bar{\Phi} = \frac{2\pi G}{3} a^2 \bar{\rho} \cdot x^2$$

$$= \frac{1}{4}\mathcal{H}^2 \Omega_{tot} \cdot x^2$$

higer-order

$$\vec{\nabla}^2 \phi = (4\pi G a^2 \bar{\rho})\,\delta$$

→ related to $\mathcal{H}$?

$$\bar{\rho}_{crit} = \frac{3H^2}{8\pi G} \Rightarrow 4\pi G = \frac{3H^2}{2\bar{\rho}_{crit}}$$

$$= \frac{3}{2} a^2 H^2 \frac{\bar{\rho}}{\bar{\rho}_{crit}} \cdot \delta = \frac{3}{2}\mathcal{H}^2 \underbrace{\Omega_{tot}}_{\sim \Omega_m} \cdot \delta$$

Euler equation (EE):

$$\frac{d\vec{v}}{dt} = -\frac{\vec{\nabla}_r p}{\rho} - \vec{\nabla}_r \Phi$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}_r \vec{v} = -\frac{\vec{\nabla}_r p}{\rho} - \vec{\nabla}_r \Phi$$

$$\left[\frac{1}{a}\frac{\partial}{\partial \tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right]\left[\mathcal{H}\vec{x} + \vec{u}\right] + (\mathcal{H}\vec{x} + \vec{u}) \cdot \frac{1}{a}\vec{\nabla}\left[\mathcal{H}\vec{x} + \vec{u}\right] = -\frac{\vec{\nabla} p}{a\bar{\rho}(1+\delta)}$$

$$- \frac{1}{a}\vec{\nabla}(\bar{\Phi} + \phi)$$

$$\left[\frac{1}{a}\frac{\partial}{\partial\tau} - \frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right](\mathcal{H}\vec{x}+\vec{u}) + \left[(\mathcal{H}\vec{x}+\vec{u})\cdot\frac{1}{a}\vec{\nabla}\right](\mathcal{H}\vec{x}+\vec{u}) = -\frac{1}{a}\nabla\Phi - \frac{1}{a}\frac{\nabla p}{\rho}$$

$$\frac{1}{a}\frac{\partial}{\partial\tau}(\mathcal{H}\vec{x}+\vec{u}) - \cancel{\left(\frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right)(\mathcal{H}\vec{x}+\vec{u})} + \cancel{\left(\frac{\mathcal{H}}{a}\vec{x}\cdot\vec{\nabla}\right)(\mathcal{H}\vec{x}+\vec{u})} \qquad (ii)$$

at constant $\vec{x}$!

$$+\left(\frac{1}{a}\vec{u}\cdot\vec{\nabla}\right)(\mathcal{H}\vec{x}+\vec{u}) = -\frac{1}{a}\vec{\nabla}\Phi - \frac{1}{a}\frac{\vec{\nabla}p}{\rho} \quad / \cdot a$$

$\vec{\nabla}\vec{x} = \mathbb{1}$ id vector

$$\frac{\partial\mathcal{H}}{\partial\tau}\vec{x} + \frac{\partial\vec{u}}{\partial\tau} + \mathcal{H}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}\Phi - \frac{\vec{\nabla}p}{\rho}$$

$$= \underbrace{-\vec{\nabla}\bar{\Phi}}_{BG} - \vec{\nabla}\phi - \frac{\vec{\nabla}p}{\rho}$$

$$\vec{x}\cdot\left[\frac{\partial\mathcal{H}}{\partial\tau} + \frac{1}{2}\mathcal{H}^2\Omega_{tot}\right] = -\frac{1}{2}\mathcal{H}^2\Omega_{tot}\cdot\vec{x} - \vec{\nabla}\phi - \frac{\vec{\nabla}p}{\rho}$$

$$= 0 \quad + \frac{\partial\vec{u}}{\partial\tau} + \mathcal{H}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}\phi - \frac{\vec{\nabla}p}{\rho}$$

$$H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{kc^2}{a^2}$$

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\bar{\rho} - kc^2 \quad / \frac{\partial}{\partial\tau}$$

$$2\mathcal{H}\frac{\partial\mathcal{H}}{\partial\tau} = \frac{8\pi G}{3}\left[\frac{\partial a^2}{\partial\tau}\bar{\rho} + a^2\frac{\partial\bar{\rho}}{\partial\tau}\right] =$$

$$= \frac{8\pi G}{3}\left[2a\frac{\partial a}{\partial\tau}\bar{\rho} + a^2\frac{\partial\bar{\rho}}{\partial\tau}\right] = \frac{8\pi G}{3}a^2\bar{\rho}\left[2\underbrace{\frac{1}{a}\frac{\partial a}{\partial\tau}}_{\mathcal{H}} + \underbrace{\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\tau}}_{-3\mathcal{H}\ \text{from BG CE}}\right]$$

$$= -\underbrace{\frac{8\pi G}{3}a^2\bar{\rho}}_{=(\mathcal{H}^2+kc^2)}\cdot\mathcal{H}$$

don't like k in eq.

$$= \mathcal{H}^2\frac{\bar{\rho}}{\rho_{crit}} = \mathcal{H}^2\Omega_{tot}$$

$$2\mathcal{H}\frac{\partial\mathcal{H}}{\partial\tau} = -\mathcal{H}^3\Omega_{tot} \Rightarrow \frac{\partial\mathcal{H}}{\partial\tau} = -\frac{1}{2}\mathcal{H}^2\Omega_{tot}$$

$$\boxed{\frac{\partial\vec{u}}{\partial\tau} + \mathcal{H}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}\phi - \cancel{\frac{\vec{\nabla}p}{\rho}} \to 0 \ \text{CDM}}$$

if: $p = p(\rho, S)$

$\Rightarrow$ new eq. for S: from $dS = \frac{dQ}{T} \Rightarrow$

$$T\frac{dS}{dt} = \frac{\mathcal{H} - \mathcal{C}}{\rho} \qquad \mathcal{H}, \mathcal{C} \text{ heating / cooling per unit vol.}$$

ideal, non-relat., monoatomic gas:

(unit mass) $TdS = d\left(\frac{3}{2}\frac{p}{\rho}\right) + p\,d\left(\frac{1}{\rho}\right)$

$$p = \frac{\rho}{\mu m_p}k_B T$$

$$\Rightarrow p \propto \rho^{5/3}e^{\frac{2}{3}\frac{\mu m_p}{k_B}S}$$

$$\frac{\vec{\nabla}p}{\bar{\rho}} = \frac{1}{\bar{\rho}}\left[\left(\frac{\partial p}{\partial\rho}\right)_S\nabla\rho + \left(\frac{\partial p}{\partial S}\right)_\rho\nabla S\right]$$

$$= c_s^2\nabla\delta + \frac{2}{3}(1+\delta)T\nabla S$$

neglect pressure:

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla}\cdot\vec{u} = 0 \qquad (CE)$$

$$\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H}\vec{u} + \underbrace{\cancel{(\vec{u}\cdot\vec{\nabla})\vec{u}}}_{\to 0 \text{ linear}} = -\vec{\nabla}\phi \qquad (EE)$$

$$\vec{\nabla}^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_{tot}\,\delta \qquad (PE)$$

$$\frac{\partial \delta}{\partial t} + \Theta = 0$$

$$\frac{\partial \Theta}{\partial \tau} = \dots$$

$\vec{\nabla}\cdot\vec{u} = \Theta$ – scalar

$$d\tau = \frac{dt}{a}$$

$$\mathcal{H} = aH = \frac{1}{a}\frac{da}{d\tau}$$

ams: $\vec{\nabla}\times(EE)$:

$$\frac{\partial}{\partial \tau}(\vec{\nabla}\times\vec{u}) = -\mathcal{H}(\vec{\nabla}\times\vec{u})$$

$$a\frac{\partial}{\partial \tau}(\vec{\nabla}\times\vec{u}) + \frac{\partial a}{\partial \tau}(\vec{\nabla}\times\vec{u}) = 0 \Rightarrow \frac{\partial}{\partial \tau}\left[a(\vec{\nabla}\times\vec{u})\right] = 0$$

$$\vec{\nabla}\times\vec{u} \propto a^{-1}$$

the real deal:

$\vec{\nabla}\cdot(EE)$:

Helmholtz: smooth, rapidly decaying vector field $=$ irrotational $(\vec{\nabla}\times = 0)$ $\vec{\nabla}\phi$ $+$ solenoidal $(\vec{\nabla}\cdot = 0)$ $\vec{\nabla}\times\vec{A}$

$$\frac{\partial\, \vec{\nabla}\cdot\vec{u}}{\partial \tau} + \mathcal{H}\cdot\vec{\nabla}\cdot\vec{u} = -\vec{\nabla}^2\phi$$

$$\frac{\partial \Theta}{\partial \tau} + \mathcal{H}\Theta = -\frac{3}{2}\mathcal{H}^2\Omega_{tot}\,\delta$$

$$-\frac{\partial^2\delta}{\partial \tau^2} - \mathcal{H}\cdot\frac{\partial \delta}{\partial \tau} = -\frac{3}{2}\mathcal{H}^2\Omega_{tot}\,\delta$$

$$\boxed{\frac{\partial^2\delta}{\partial \tau^2} + \underbrace{\mathcal{H}\frac{\partial \delta}{\partial \tau}}_{\text{Hubble exp.}} - \underbrace{\frac{3}{2}\mathcal{H}^2\Omega_{tot}\,\delta}_{\text{gravity}} = 0}$$

$\delta = \delta(\vec{x},\tau)$!

go to Fourier space: $\delta \to \delta_k(\vec{k},\tau)$

$$\frac{\partial^2\delta_k}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_k}{\partial \tau} - \frac{3}{2}\mathcal{H}^2\Omega_{tot}\,\delta_k = 0$$

decompose: $\delta_k = A_k(\vec{k})\cdot \underbrace{D(\tau)}_{\text{growth factor}}$ true for linear

$$\boxed{\frac{\partial^2 D}{\partial \tau^2} + \mathcal{H}\frac{\partial D}{\partial \tau} - \frac{3}{2}\mathcal{H}^2\Omega_{tot}\,D = 0}$$

(6)

Matter dominated era : + flat universe ($k=0$)

$$\left[H^2 = \frac{8\pi G}{3}\bar{\rho}_m \right.$$

continuity eq.

$$\frac{\partial \bar{\rho}}{\partial \tau} + 3\mathcal{H}\bar{\rho} = 0 \Rightarrow \frac{\partial \bar{\rho}}{\partial \tau} + 3\frac{1}{a}\frac{\partial a}{\partial \tau}\bar{\rho} = 0$$

$$\frac{\partial \ln \bar{\rho}}{\partial \tau} = -3\frac{\partial \ln a}{\partial \tau} \quad \Big/ \int d\tau$$

$$\bar{\rho}_m = \bar{\rho}_{crit.} \cdot \Omega_m(t_0)\, a^{-3} \qquad \ln \bar{\rho}_0 - \ln \bar{\rho}_a = -3(\underbrace{\ln a_0^{=1}}_{=0} - \ln a)$$

$$\ln \frac{\bar{\rho}_0}{\bar{\rho}} = 3 \ln a$$

$$\left. \bar{\rho}/\bar{\rho}_0 = a^{-3} \right]$$

$$\left\{ \begin{array}{l} \Omega_{tot} = \Omega_m = \boxed{1 \quad EdS} \\ \mathcal{H}' = -\frac{1}{2}\mathcal{H}^2 \Rightarrow \boxed{\mathcal{H} = \frac{2}{\tau}} \qquad \left(\mathcal{H}' = -\frac{2}{\tau^2},\ \frac{1}{2}\mathcal{H}^2 = \frac{1}{2}4/\tau^2 = 2/\tau^2\right) \\ \mathcal{H} = \frac{1}{a}\frac{\partial a}{\partial \tau} \Rightarrow a \propto \tau^2 \end{array} \right.$$

$$\frac{\partial^2 D}{\partial \tau^2} + \overset{\mathcal{H}}{\left(\frac{2}{\tau}\right)}\frac{\partial D}{\partial \tau} - \frac{3}{2}\overset{\mathcal{H}^2}{\left(\frac{4}{\tau^2}\right)}\overset{=\Omega_m}{(1)} D = 0$$

$$\frac{\partial^2 D}{\partial \tau^2} + \frac{2}{\tau}\frac{\partial D}{\partial \tau} - 6/\tau^2 \cdot D = 0$$

try: $D = c_1 a^n = c_1 \cdot a^n \qquad c_1$ - const.

$$\frac{\partial a^n}{\partial \tau} = n a^{n-1}\frac{\partial a}{\partial \tau} = n a^n \mathcal{H}$$

$$\frac{\partial^2 a^n}{\partial \tau^2} = n\left[\frac{\partial a^n}{\partial \tau}\mathcal{H} + a^n \frac{\partial \mathcal{H}}{\partial \tau}\right] = n\left[n a^n \mathcal{H}^2 - \frac{1}{2}a^n \mathcal{H}^2\right]$$

$$= a^n \mathcal{H}^2\left[n^2 - \frac{1}{2}n\right]$$

$$a^n \mathcal{H}^2\left[n^2 - \frac{1}{2}n\right] + \mathcal{H} \cdot n a^n \mathcal{H} - \frac{3}{2}\mathcal{H}^2 a^n = 0$$

$$n^2 - \frac{1}{2}n + n - \frac{3}{2} = 0 \Rightarrow n^2 + \frac{1}{2}n - \frac{3}{2} = 0$$

$$\left(n + \frac{3}{2}\right)(n-1) = 0$$

$$D(\tau) \propto \begin{cases} a & \text{growing mode} \quad D_+ \\ a^{-3/2} & \text{decaying mode} \quad D_- \end{cases}$$

in principle I have both:

$$\delta(\vec{k},\tau) = A_k \cdot a + B_k \cdot a^{-3/2}$$

what is the velocity:

$$\vec{\nabla}\cdot\vec{u} = \Theta$$

from the continuity eq.

$$\Theta = -\frac{\partial \delta}{\partial \tau} \Rightarrow FT \Rightarrow \Theta_k = -\frac{\partial \delta_k}{\partial \tau}$$

$$\Theta_k = -A_k \frac{\partial a}{\partial \tau} + (-B_k)\left(-\frac{3}{2}\right) a^{-5/2} \frac{\partial a}{\partial \tau} =$$

$$= -\left[A_k\, a\mathcal{H} + B_k\left(-\frac{3}{2}\right) a^{-3/2}\mathcal{H}\right] =$$

$$= -\mathcal{H}\left[A_k \cdot a - \frac{3}{2} B_k a^{-3/2}\right]$$

if: $A_k > 0$ and $B_k = 0$

$\rightarrow \delta_k > 0$ so overdensity with $\Theta_k < 0$ vel. going towards the high density region

$A_k = 0$ and $B_k > 0$

$\delta_k > 0$ overdensity with $\Theta_k > 0$ going away from high dens. regions.

DM perturbations grow as $\propto a$ during MD era

Radiation dominated era:

go back a step:

$$\vec{\nabla}^2 \phi = 4\pi G a^2 \underbrace{\sum_i \bar{\rho}_i \delta_i}_{= \bar{\rho}_r \delta_r + \bar{\rho}_m \delta_m}$$

all the contributions

baryons + photons (radiation) coupled $\left(\dot{\Theta}_{r,0} = \frac{1}{3}\dot{\delta}^0\right)$

write down fluid eq.: for baryons

sound speed

$$0 = \frac{\partial^2 \delta_b}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_b}{\partial \tau} - \underbrace{\frac{3}{2}\mathcal{H}^2 \Omega_{tot}\delta}_{\vec{\nabla}^2 \phi_{dm} = k^2 \phi_{dm}(\vec{k})} + k^2 c_s^2 \delta_b = 0$$

but not baryons

$\rho_{dm} \gg \rho_b$

$$\frac{\vec{\nabla} p}{\rho} = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)\vec{\nabla}\rho = \frac{1}{\rho}\underbrace{\left(\frac{\partial p}{\partial \rho}\right)}_{c_s^2}\cdot \rho\,\vec{\nabla}\delta$$

$$\frac{\partial^2 \delta_b}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_b}{\partial \tau} + k^2 c_s^2 \delta_b = k^2 \phi_{dm} \approx \text{const}$$

$$\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H}\vec{u} = -\vec{\nabla}\phi - \frac{\vec{\nabla} p}{\rho} = -\vec{\nabla}\phi - c_s^2 \vec{\nabla}\delta$$

$$\frac{\partial \Theta}{\partial \tau} + \mathcal{H}\Theta = -\underbrace{\vec{\nabla}^2\phi}_{k^2\phi} - c_s^2 \underbrace{\vec{\nabla}^2\delta}_{k^2\delta}$$

$\sum_i \bar\rho_i \delta_i = \bar\rho_r \delta_r + \bar\rho_m \delta_m$ (hand-waving) (8)

↳ during RD δ_r is oscilating for scales much smaller than horizon ($kH \gg 1$)

time averaged density is only DM

TO PROVE RIGOROUSLY NEED REL. PERT. THEORY

$\Rightarrow$ CDM the only clustered component during A Osc. of R

$$\frac{\partial^2 \delta_m}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_m}{\partial \tau} - 4\pi G a^2 \bar\rho_m \delta_m = 0$$

$\mathcal{H}^2 \simeq \frac{8\pi G}{3} \bar\rho_r a^2$ hom. part is the largest in $\bar\rho_r \gg \bar\rho_m$

but δ_r oscil.

show: $\cancel{\mathcal{H}^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^4}}$

$\cancel{\frac{1}{a}\frac{\partial a}{\partial \tau} = A/a^2}$

$\cancel{\frac{a\,\partial a}{\partial \tau} = A \Rightarrow a \propto t^{1/2}}$ $\Rightarrow \cancel{\mathcal{H} = \frac{1}{a}\frac{\partial a}{\partial \tau} = \frac{1}{2\tau}}$

$\cancel{\frac{\partial (a^2)}{\partial \tau} \propto \text{const.}}$

$$\mathcal{H}^2 \simeq \frac{8\pi G}{3} a^2 \bar\rho_r \propto 1/a^2$$

$\mathcal{H} \propto 1/a$

$\frac{1}{a}\frac{\partial a}{\partial \tau} \propto 1/a \Rightarrow \frac{\partial a}{\partial \tau} \propto$ const $\quad a \propto \tau \Rightarrow \mathcal{H} = \frac{1}{\tau}$

$$\frac{1}{\mathcal{H}^2}\frac{\partial^2 \delta_m}{\partial \tau^2} + \frac{1}{\mathcal{H}}\frac{\partial \delta_m}{\partial \tau} - \underbrace{\frac{4\pi G a^2 \bar\rho_m}{\mathcal{H}^2}}\delta_m = 0$$

$\cancel{dy = \mathcal{H} d\tau}$

$\cancel{\frac{\partial^2 \delta_m}{\partial y^2}}$ $\quad \frac{4\pi G a^2 \bar\rho_m}{\frac{8\pi G}{3} a^2 \bar\rho_r}\delta_m \sim 0$

since δ_m is evolving only on cosmological scales (i.e. no pressure to have other time scale)

$$\frac{\partial^2 \delta_m}{\partial \tau^2} \sim \mathcal{H}^2 \delta_m \gg 4\pi G \bar\rho_m a^2 \delta_m$$

$\bar\rho_r \gg \bar\rho_m$

$$\frac{\partial^2 \delta_m}{\partial \tau^2} + \mathcal{H}\frac{\partial \delta_m}{\partial \tau} = 0 \Rightarrow \delta_m \propto \text{const.}$$

$\delta_m \propto \ln\tau \propto \ln a \quad \Rightarrow \quad \frac{\partial \delta_m}{\partial \tau} \propto \frac{1}{\tau} \quad \frac{\partial^2 \delta_m}{\partial \tau^2} \propto -\frac{1}{\tau^2}$

$-\frac{A}{\tau^2} + \frac{A}{\tau^2} = 0 \quad \cancel{\text{if } A} \quad A \neq 0$