Cosmology and the Large Scale Structure of the Universe

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Addendum

\[ m - M = 5 \log_{10} D_L - 1 \]

\[ D_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} \]

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Figure 2. (a) Hubble diagram for 42 high-redshift type Ia supernovae from the Supernova Cosmology Project and 18 low-redshift type Ia supernovae from the Supernova Survey, plotted on a linear redshift scale to display details at high redshift. The symbols and curves are as in Fig. 1.

(b) Uncertainty-normalized residuals from the best-fitting cosmology for the supernovae subset, \((0.28, 0.72)\). The dashed curves are for a range of cosmological models: on top, \((0.5, 0.5)\) third from bottom, \((0.75, 0.25)\) second from bottom, and \((1, 0)\) is the solid curve on bottom. The middle solid curve is for the cosmological models with \((c, \sigma_c)\).

(c) Magnitude residuals from the best-fitting cosmology for the supernovae subset, \((0.73, 1.32)\). The dashed curves are for a range of cosmological models: on top, \((0.5, 0.5)\) third from bottom, \((0.75, 0.25)\) second from bottom, and \((1, 0)\) is the solid curve on bottom. The middle solid curve is for the cosmological models with \((0, 1)\).

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Note that this plot is practically identical to the magnitude residual plot for the best-fitting unconstrained cosmology. We have compared the results of Bayesian and classical, frequentist, fitting procedures. For the Bayesian fits, we have assumed a prior probability distribution that has zero probability for but otherwise has uniform probability in the four parameters \(a, b, M, M_B\).

For the frequentist fits, we have followed the classical statistical procedures described by Feldman & Cousins (1998) to guarantee frequentist coverage of our confidence regions in the physically allowed part of parameter space. Note that throughout the previous cosmology literature, completely different conventions have been used to define the confidence regions and the sensitivity to the various cosmological parameters.

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[Perlmutter et al (1999)]
Recap

The galaxy 2-point function is the excess probability of finding two galaxies in the volume elements $dV_1$ and $dV_2$

$$dP = dV_1 \, dV_2 \left< n_g(\vec{x}_1) \, n_g(\vec{x}_2) \right> = dV_1 \, dV_2 \, \bar{n}_g^2 \left[ 1 + \xi(r) \right]$$

where $n_g(\vec{x}) \equiv \bar{n}_g \left[ 1 + \delta_g(\vec{x}) \right]$

\[\xi(r) \gg 1\]
Recap

The power spectrum is a measure of the amplitude of perturbations as a function of scale … … and the Fourier Transform of the 2PCF

\[ P(k) = \int \frac{d^3 x}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \xi(x) \]

The power spectrum is what we want to predict
Today's goal

Goal:
predict the correlation functions describing the statistical properties of the Large-Scale Structure

for this we study the evolution of

\[ \delta_{\vec{k}}(t) \]

We need:
1. Equations of motion
2. Initial conditions
Evolution of matter perturbation:
Initial Conditions
Part 1: Inflation
The Horizon Problem:
In the CMB we observe a large number ($\sim 10^4$) of causally disconnected patches … all at the same temperature!
The curvature contribution at present time is small …

implying that in the Early Universe should be extremely (i.e. unnaturally) small!

\[ |1 - \Omega| = \left| \frac{k}{a^2 \rho_c} \right| \simeq 0 \]

but … \( \sim a^2 \)

**The Flatness Problem:**
The curvature contribution at present time is small …

implying that in the Early Universe should be extremely (i.e. unnaturally) small!

\[ |1 - \Omega(t_{BBN})| \lesssim 10^{-6} \]
Unwanted Relics:
Grand Unified Theories (GUTs) predict an overabundance of topological defects (e.g. magnetic monopoles) from phase transitions in the Early Universe … but we don’t seen any of such things!
The \textit{inflationary} solution

Horizon, flatness, unwanted relics …

Guth’s idea, 1980: \textit{Inflation} can solve \textit{all} these problems at once!

The Universe underwent a period of \textit{accelerated expansion} in its early history

NB: this is not a “theory”, nor a “model”
Hubble’s law $v = H d$ can also be written like this: $d = \frac{v}{H}$

The distance $d$ where the velocity $v$ equals the speed of light is the “Hubble horizon”
Solving the horizon problem

Inflation

observable Universe

horizon \((H^{-1})\)

\[ 10^{-28} \text{ cm} \]

\[ v = H d \]
Solving the horizon problem

Inflation

observable Universe

horizon \((H^{-1})\)
Solving the horizon problem

Inflation

observable Universe

horizon ($H^{-1}$)
Solving the horizon problem

Inflation

observable Universe

horizon \( (H^{-1}) \)

\sim 1 \text{ cm}
Solving the horizon problem

Radiation domination

observable Universe

horizon ($H^{-1}$)
Solving the horizon problem

observable Universe

horizon ($H^{-1}$)

Radiation domination
Solving the horizon problem

Matter domination

observable Universe

horizon ($H^{-1}$)
Solving the horizon problem

All points in the observable Universe have been in causal contact at some previous time

(ergo, the isotropy of the CMB)
Solving the flatness problem

A curved space-time would be flattened as an “exponentially inflated balloon”

observable Universe

horizon \( (H^{-1}) \)

\(~ 10^{28} \text{ cm} \)
Getting rid of GUT relics

The number density of unwanted relics would also be damped to negligible values.

Today

observable Universe

horizon ($H^{-1}$)

~ $10^{28}$ cm
Energy content

How do we get acceleration?

\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho \]

\[ \frac{\ddot{a}^2}{a^2} = -\frac{4\pi G}{3} (\rho + 3p) \]

We need something with “negative pressure” …

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We need something with “negative pressure” …

\[ \phi(\vec{x}, t) = \phi_0(t) + \delta \phi(\vec{x}, t) \quad V(\phi_0) \]

A scalar field … with some potential

\[ \rho = \frac{1}{2} \dot{\phi}_0^2 + V(\phi_0) \]

\[ p = \frac{1}{2} \dot{\phi}_0^2 - V(\phi_0) \]

Slow Roll Inflation

How do we get acceleration?

\[
\frac{\ddot{a}^2}{a^2} = \frac{8\pi G}{3}\rho
\]

\[
\frac{\ddot{a}^2}{a^2} = -\frac{4\pi G}{3} \left( \rho + 3p \right)
\]

We need something with “negative pressure” ...

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\phi(\mathbf{x}, t) = \phi_0(t) + \delta \phi(\mathbf{x}, t)
\]

\[\begin{align*}
\rho &= \frac{1}{2} \dot{\phi}_0^2 + V(\phi_0) \\
p &= \frac{1}{2} \dot{\phi}_0^2 - V(\phi_0)
\end{align*}\]

A scalar field ... with some potential

flat potential

\[V'(\phi) \equiv \frac{dV}{d\phi} \simeq 0\]

\[\frac{1}{2} \dot{\phi}_0^2 \simeq 0\]
Slow Roll Inflation

How do we get acceleration?

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho
\]

\[
\frac{\ddot{a}^2}{a^2} = -\frac{4\pi G}{3} (\rho + 3p)
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We need something with “negative pressure” …

\[
\phi(\vec{x}, t) = \phi_0(t) + \delta \phi(\vec{x}, t)
\]

\[
V(\phi_0)
\]

\[
\begin{cases}
\rho \approx +V(\phi_0) \\
p \approx -V(\phi_0)
\end{cases}
\]

A scalar field … with some potential

the potential energy dominates over the kinetic one, then

How do we get acceleration?

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We need something with “negative pressure” ...

\[ \phi(\vec{x}, t) = \phi_0(t) + \delta\phi(\vec{x}, t) \quad V(\phi_0) \]

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A scalar field ... with some potential

the potential energy dominates over the kinetic one, then

\[ H = \frac{\dot{a}}{a} \simeq \left[ \frac{8\pi G}{3}V(\phi_0) \right]^{1/2} \sim \text{constant} \]

\[ a(t) \sim e^{Ht} \]

exponential expansion!
SO …

The horizon problem, the flatness problem are solved while unwanted relics are swept away …

… but these are “theorists’ problems”: inflation does not solve any tension of the theory with the data!
The horizon problem, the flatness problem are solved while unwanted relics are swept away …

… but these are “theorists’ problems”: inflation does not solve any tension of the theory with the data!

Moreover, there is plenty of models implementing slow-roll inflation (and other varieties of inflation), covering a wide range of energy scales.
The horizon problem, the flatness problem are solved while unwanted relics are swept away …

… but these are “theorists’ problems”: inflation does not solve any tension of the theory with the data!

Moreover, there is plenty of models implementing slow-roll inflation (and other varieties of inflation), covering a wide range of energy scales

Still … Inflation provides two, crucial, unrequested predictions:

density perturbations and gravitational waves
Density Perturbations from Inflation

\[ \phi(\vec{x}, t) = \phi_0(t) + \delta \phi(\vec{x}, t) \]  
\( \text{quantum fluctuations of the inflaton} \)

for slow-roll inflation, their equation of motion in Fourier space is

\[ \ddot{\delta \phi_{\vec{k}}} + 2aH \dot{\delta \phi_{\vec{k}}} + k^2 \delta \phi_{\vec{k}} = 0 \]  
\( \text{quantum harmonic oscillator for } k \gtrsim aH \)

one can compute the \textbf{power spectrum of inflaton fluctuations}

\[ P_{\delta \phi}(k) \equiv |\delta \phi_{\vec{k}}|^2 = \frac{H^2}{2k^3} \]  

\[ \text{quantum fluctuations} \]  
\[ 10^{-28} \text{ cm} \]  

\[ \text{"frozen" curvature perturbations} \]

\[ \text{observed density perturbations} \]  
\[ \text{Inflation} \]  
\[ \text{Today} \]
Density Perturbations from Inflation

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\[ P_{\delta \phi}(k) \equiv |\delta \phi_k|^2 = \frac{H^2}{2k^3} \]

and (after a lot of pain) the power spectrum of the gravitational potential perturbations as they enter the horizon again

\[ P_{\Phi}(k) = \frac{2}{9M_p^4} \frac{V^2}{V'^2} \frac{H^2}{k^3} \bigg|_{aH=k} \]

\[ M_p = \frac{1}{\sqrt{8\pi G}} \]

observed density perturbations

Today
Density Perturbations from Inflation

Inflation predicts the power spectrum of the perturbations in the gravitational potential (and in the energy density) today!

\[ P_\Phi(k) = \frac{2}{9 M_p^4} \frac{H^2 V^2}{V'^2} k^{-4+n_s} \bigg|_{aH=k} \]

- **Amplitude**: \( \Delta_\Phi(k) \equiv 4\pi k^3 P_\Phi(k) \sim \text{constant} \sim (10^{-5})^2 \)
- **Spectral index**: \( n_s = 1 - 2M_p^2 \left( \frac{V'}{V} \right)^2 + 2M_p^2 \frac{V''}{V} \)
- **Scale-dependence**
Evolution of matter perturbations:
Initial Conditions
Part 2: From inflation to photon decoupling
The evolution of density perturbations before decoupling

The “linear size” (not the amplitude!!) of a perturbation grows with the scale factor (by definition of scale factor …)
The evolution of density perturbations before decoupling

\[ \lambda = a \lambda_{\text{com}} \]

We can observe today (with CMBB or LSS) perturbations over a finite range of scales.
The evolution of density perturbations before decoupling

in order to study the evolution of perturbation we need to compare their linear size to the Hubble horizon at any given time
The evolution of density perturbations before decoupling

Super-horizon curvature perturbations are “frozen”
The evolution of density perturbations before decoupling

Perturbations at different scales have a different sub-horizon evolution: perturbations (in the potential) already sub-horizon during radiation domination are suppressed.
My initial conditions

- Inflation
- Radiation
- Matter
- Dark energy

Time, $\ln a$

Physical distance, $\ln \lambda$

- Matter/radiation eq.
- Photon decoupling / CMB
- Last scattering surface
- Today
My initial conditions

- Time, ln a
- Physical distance, ln λ

- Inflation
- Radiation
- Matter
- Dark energy

- Matter/radiation eq.
- Photon decoupling / CMB
- Last scattering surface
- Today
The “initial” matter power spectrum

Let’s consider a scale (mode $k$) that re-enters the horizon during matter domination (that is a large scale today!)

$$\Delta \Phi(k) \equiv 4\pi k^3 P_\Phi(k) \simeq \text{constant} \quad P_\Phi(k) \simeq \frac{C}{k^3}$$

To obtain the matter power spectrum I should relate matter and gravitational potential perturbations via Poisson’s equation

$$\nabla_r^2 \Phi_{tot} = 4\pi G \rho$$

$$\rho(\vec{r}, t) = \bar{\rho}(t) + \delta \rho(\vec{r}, t)$$

$$\Phi_{tot}(\vec{r}, t) = \bar{\Phi}(\vec{r}, t) + \Phi(\vec{r}, t)$$

$$\nabla_r^2 \bar{\Phi} = 4\pi G \bar{\rho} \quad \rightarrow \quad \bar{\Phi} = \frac{2\pi G}{3} r^2 \bar{\rho}$$

“Jean’s swindle”

$$\nabla_r^2 \Phi(\vec{r}, t) = 4\pi G \delta \rho(\vec{r}, t)$$

in comoving coordinates $\vec{r} = a(t) \vec{x}$

$$-k^2 \Phi_\vec{k} = 4\pi G a^2 \bar{\rho} \delta_\vec{k}$$

$$\langle |\delta_\vec{k}|^2 \rangle \sim k^4 \langle |\Phi_\vec{k}|^2 \rangle$$
The "initial" matter power spectrum

The linear matter power spectrum at $z \approx 1000$

$$P(k) \sim \frac{k^4}{C}$$

Primordial scale-invariant power spectrum

Suppression due to radiation pressure

$P(k) \sim k^{-2.5}$
The “initial” matter power spectrum

The linear matter power spectrum at $z \approx 1000$

$$P(k) \sim k^4 \quad P_{\Phi}(k) \sim Ck \, T^2(k)$$
The “initial” matter power spectrum

The linear matter power spectrum at $z \approx 1000$

$$P(k) \sim k^4 P_\Phi(k) \sim C k T^2(k)$$
Evolution of matter perturbations:
Equations of motion
Evolution of matter perturbations

We will consider now the following approximations for the evolution of matter perturbations:

1. **All matter is cold** (ignore the effects of baryons & neutrinos)

2. **Newtonian approximation:**
   \[ k \gg a H(a) \] scales much smaller than the horizon
   \[ v \ll c \] velocities much smaller than the speed of light

3. **Matter domination** (ignore effects of dark energy at late times)
Evolution of matter perturbations

Cold Dark Matter

Warm Dark Matter
Evolution of matter perturbations

**Cold Dark Matter**

**Warm Dark Matter**

![Graph showing the evolution of matter perturbations for Cold and Warm Dark Matter with a logarithmic scale for $P_L(k)$ and $k$ [h Mpc$^{-1}$].]
Fluid equations

Assuming **CDM as ideal fluid** we need the following equations:

**Continuity equation (conservation of mass)**

\[
\frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho \vec{v}) = 0
\]

**Euler’s equation (conservation of momentum)**

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla_r) \vec{v} = -\frac{\nabla \rho}{\rho} - \nabla_r \Phi_{tot}
\]

- Pressure term (vanishing for CDM)
- Force

**Poisson’s equation**

\[
\nabla_r^2 \Phi_{tot} = 4\pi G \rho
\]

3 equations, 3 unknowns: \( \rho , v \) and \( \Phi_{tot} \)
for Cold Dark Matter we can ignore the thermal motion of individual particles, and study the evolution of perturbations.
We want the equations of motions for *perturbations* and as a function of comoving coordinates and conformal time

\[ d\tau = \frac{dt}{a(t)} \]

For the *matter density* we have

\[ \rho(\vec{x}, \tau) = \bar{\rho}(\tau)[1 + \delta(\vec{x}, \tau)] \]

\[ \delta(\vec{x}, \tau) \quad \text{matter perturbations} \]
Perturbations

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For the **matter velocity**, instead we have

\[ \vec{r} = a(t) \vec{x} \quad \vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{da}{dt} \vec{x} + a \frac{d\vec{x}}{dt} = H(\tau) \vec{r}(\tau) + \vec{u}(\vec{x}, \tau) \]

\[ \mathcal{H} \equiv \frac{1}{a} \frac{da}{d\tau} = a \mathcal{H} \]

Hubble flow

\[ \vec{v}(\vec{x}, \tau) = \mathcal{H}(\tau) \vec{x}(\tau) + \vec{u}(\vec{x}, \tau) \]

\[ \vec{u}(\vec{x}, \tau) \quad \text{peculiar velocities} \]
Perturbations

We want the equations of motions for perturbations and as a function of comoving coordinates and conformal time

\[ d\tau = \frac{dt}{a(t)} \]

For the matter density we have

\[ \rho(\vec{x}, \tau) = \bar{\rho}(\tau) [1 + \delta(\vec{x}, \tau)] \]

\[ \delta(\vec{x}, \tau) \] matter perturbations

For the matter velocity, instead we have

\[ \vec{r} = a(t) \vec{x} \]

\[ \vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{da}{dt} \vec{x} + a \frac{d\vec{x}}{dt} = H(\tau) \vec{r}(\tau) + \vec{u}(\vec{x}, \tau) \]

\[ H \equiv \frac{1}{a} \frac{da}{d\tau} = a H \] Hubble flow

\[ \vec{v}(\vec{x}, \tau) = \mathcal{H}(\tau) \vec{x}(\tau) + \vec{u}(\vec{x}, \tau) \]

\[ \vec{u}(\vec{x}, \tau) \] peculiar velocities

\[ \Phi_{tot}(\vec{x}, \tau) = \bar{\Phi}(\vec{x}, \tau) + \Phi(\vec{x}, \tau) \]

\[ \Phi(\vec{x}, \tau) \] gravitational potential perturbations
Equations for the perturbations

Assuming CDM as ideal fluid we need the following equations:

\[
\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1 + \delta) \vec{u}] = 0
\]
continuity equation

\[
\frac{\partial \vec{u}}{\partial \tau} + \mathcal{H} \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \Phi
\]
Euler’s equation

\[
\nabla^2 \Phi = 4\pi G \bar{\rho} a^2 \delta \quad \text{but from Friedmann's eq.} \quad \mathcal{H}^2 = \frac{8\pi G}{3} a^2 \bar{\rho}
\]

\[
\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta
\]
Poisson’s equation

Again: 3 equations, 3 unknowns: \( \delta \), \( \vec{u} \) and \( \Phi \)
Equations for the perturbations

Linearizing …

\[ \frac{\partial \delta}{\partial \tau} + \bar{\nabla} \cdot [(1 + \delta) \bar{u}] = 0 \]

... continuity equation

\[ \frac{\partial \bar{u}}{\partial \tau} + \mathcal{H} \bar{u} + (\bar{u} \cdot \bar{\nabla}) \bar{u} = -\bar{\nabla} \Phi \]

... Euler’s equation

\[ \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta \]

... Poisson’s equation
Equations for the perturbations

Linearizing ...

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\[ \frac{\partial \vec{u}}{\partial \tau} + \mathcal{H} \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \Phi \]
Euler’s equation

\[ \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta \]
Poisson’s equation
Equations for the perturbations

Linearizing …

\[ \frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot \vec{u} = 0 \]  
continuity equation

\[ \frac{\partial \vec{u}}{\partial \tau} + \mathcal{H} \vec{u} = -\vec{\nabla} \Phi \]  
Euler’s equation

\[ \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta \]  
Poisson’s equation
Equations for the perturbations

Linearizing ...

\[ \frac{\partial \delta}{\partial \tau} + \nabla \cdot \bar{u} = 0 \]

continuity equation

\[ \nabla \cdot \left( \frac{\partial \bar{u}}{\partial \tau} + \mathcal{H} \bar{u} = -\nabla \Phi \right) \]

Euler’s equation

\[ \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \delta \]

Poisson’s equation

then introducing the velocity divergence

\[ \theta(x, \tau) \equiv \nabla \cdot \bar{u}(x, \tau) \]
Linear equations for the perturbations

\[ \frac{\partial \delta}{\partial \tau} + \theta = 0 \]
continuity equation

\[ \frac{\partial \theta}{\partial \tau} + \mathcal{H} \theta + \frac{3}{2} \mathcal{H}^2 \delta = 0 \]
Euler’s equation

\[ \frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta}{\partial \tau} - \frac{3}{2} \mathcal{H}^2 \delta = 0 \]
2nd order equation

where (for a flat, matter-dominated Universe)
\[ \mathcal{H} = \frac{1}{a} \frac{da}{d\tau} = \frac{2}{\tau} \]
Linear growth of perturbations

\[ \frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} - \frac{3}{2} \mathcal{H}^2 \delta_k = 0 \]

2nd order equation in Fourier space
Linear growth of perturbations

\[ \frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} - \frac{3}{2} \mathcal{H}^2 \delta_k = 0 \]

2nd order equation in Fourier space

Look for a separable solution like \( \delta_k(\tau) = D(\tau) A_k \)

\( D(\tau) \) growth factor

\[ \begin{align*}
D_+(a) & \sim a & \text{growing mode} \\
D_-(a) & \sim a^{-3/2} & \text{decaying mode}
\end{align*} \]

\( \delta_k(a) = A_k a + B_k a^{-3/2} \)

\( \theta_k(a) = - \frac{\partial \delta_k}{\partial \tau} = -\mathcal{H} \left( A_k a - \frac{3}{2} B_k a^{-3/2} \right) \)
Linear growth of perturbations

\[ \frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} - \frac{3}{2} \mathcal{H}^2 \delta_k = 0 \]

2nd order equation in Fourier space

Look for a separable solution like \( \delta_k(\tau) = D(\tau) A_k \)

\( D(\tau) \) growth factor

\[
\begin{cases}
    D_+(a) \sim a & \text{growing mode} \\
    D_-(a) \sim a^{-3/2} & \text{decaying mode}
\end{cases}
\]

\[
\delta_k(a) = A_k^+ a + B_k^- a^{-3/2}
\]

\[
\theta_k(a) = - \frac{\partial \delta_k}{\partial \tau} = -\mathcal{H} \left( A_k^+ a - \frac{3}{2} B_k^- a^{-3/2} \right)
\]

Growing mode:
- \( A_k^+ \neq 0 \)
- \( B_k^- = 0 \)
- \( \delta > 0 \)
- \( \theta < 0 \)

Decaying mode:
- \( A_k^- = 0 \)
- \( B_k^- \neq 0 \)
- \( \delta > 0 \)
- \( \theta > 0 \)
Linear growth in a $\Lambda$CDM cosmology

\[ (\Omega_m = 1, \ \Omega_\Lambda = 0) \]

\[ (\Omega_m = 0.3, \ \Omega_\Lambda = 0.7) \]
Linear growth in a flat, $\Lambda$CDM cosmology

$\Omega_m = 1, \Omega_\Lambda = 0$  

$\Omega_m = 0.3, \Omega_\Lambda = 0.7$

growth rate

$$f \equiv \frac{d \ln D}{d \ln a} \simeq \Omega_m^{0.55} (z)$$
Linear vs Nonlinear evolution

\[ \langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) \, P(k_1) \]

\[ P_L(k, a) = D^2(a) \, P_0(k) \]
Linear vs Nonlinear evolution

\[
\langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) P(k_1)
\]

\[
P_L(k, a) = D^2(a) P_0(k)
\]

\[z = 2\]
\[z = 10\]
Linear vs Nonlinear evolution

\[ D^2(a) \]

\[ \langle \delta_{k_1} \delta_{k_2} \rangle = \delta_D(k_1 + k_2) P(k_1) \]

\[ P_L(k, a) = D^2(a) P_0(k) \]

\[ z = 1 \]
\[ z = 2 \]
\[ z = 10 \]
Linear vs Nonlinear evolution

\[ \langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) P(k_1) \]

\[ P_L(k, a) = D^2(a) P_0(k) \]

\[ z = 0.5 \]
\[ z = 1 \]
\[ z = 2 \]
\[ z = 10 \]
Linear vs Nonlinear evolution

\[ \langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \rangle = \delta_D(\vec{k}_1 + \vec{k}_2) P(k_1) \]

\[ P_L(k, a) = D^2(a) P_0(k) \]

\[ z = 0 \]
\[ z = 0.5 \]
\[ z = 1 \]
\[ z = 2 \]
\[ z = 10 \]
Linear vs Nonlinear evolution

\[ \langle \delta_{k_1} \delta_{k_2} \rangle = \delta_D (k_1 + k_2) P(k_1) \]

\[ P_L(k, a) = D^2(a) P_0(k) \]

\begin{align*}
  z &= 0 \\
  z &= 0.5 \\
  z &= 1 \\
  z &= 2 \\
  z &= 10
\end{align*}
Linear vs Nonlinear evolution

\[ P_{NL}(k) = P_L(k) + \Delta P_{NL}(k) \]

nonlinear corrections!
$P_{NL}(k) = P_L(k) + \Delta P_{NL}(k)$

Nonlinear corrections!
Linear vs Nonlinear evolution

\[ P_{NL}(k) = P_L(k) + \Delta P_{NL}(k) \]

nonlinear corrections!
Linear vs Nonlinear evolution

\[ P_{NL}(k) = P_L(k) + \Delta P_{NL}(k) \]

nonlinear corrections!
Linear vs Nonlinear evolution

\[ P_{NL}(k) = P_{L}(k) + \Delta P_{NL}(k) \]

nonlinear corrections!
Linear vs Nonlinear evolution

\[ P_{NL}(k) = P_L(k) + \Delta P_{NL}(k) \]

nonlinear corrections!

This is a proof of Dark Matter!
The growth of matter perturbations

\[ \Delta(k) = 4\pi k^3 P(k) \]
The growth of matter perturbations

Linear & mildly nonlinear regime:
Analytical, Perturbation Theory

\[ P(k, z) = D_+^2 z P_0(k) + P_{1\text{loop}}(k, z) + P_{2\text{loop}}(k, z) + \ldots \]

\[ \Delta(k) \equiv 4\pi k^3 P(k) \]

Nonlinear regime:
Phenomenological models, N-body simulations

Accurate predictions are crucial!