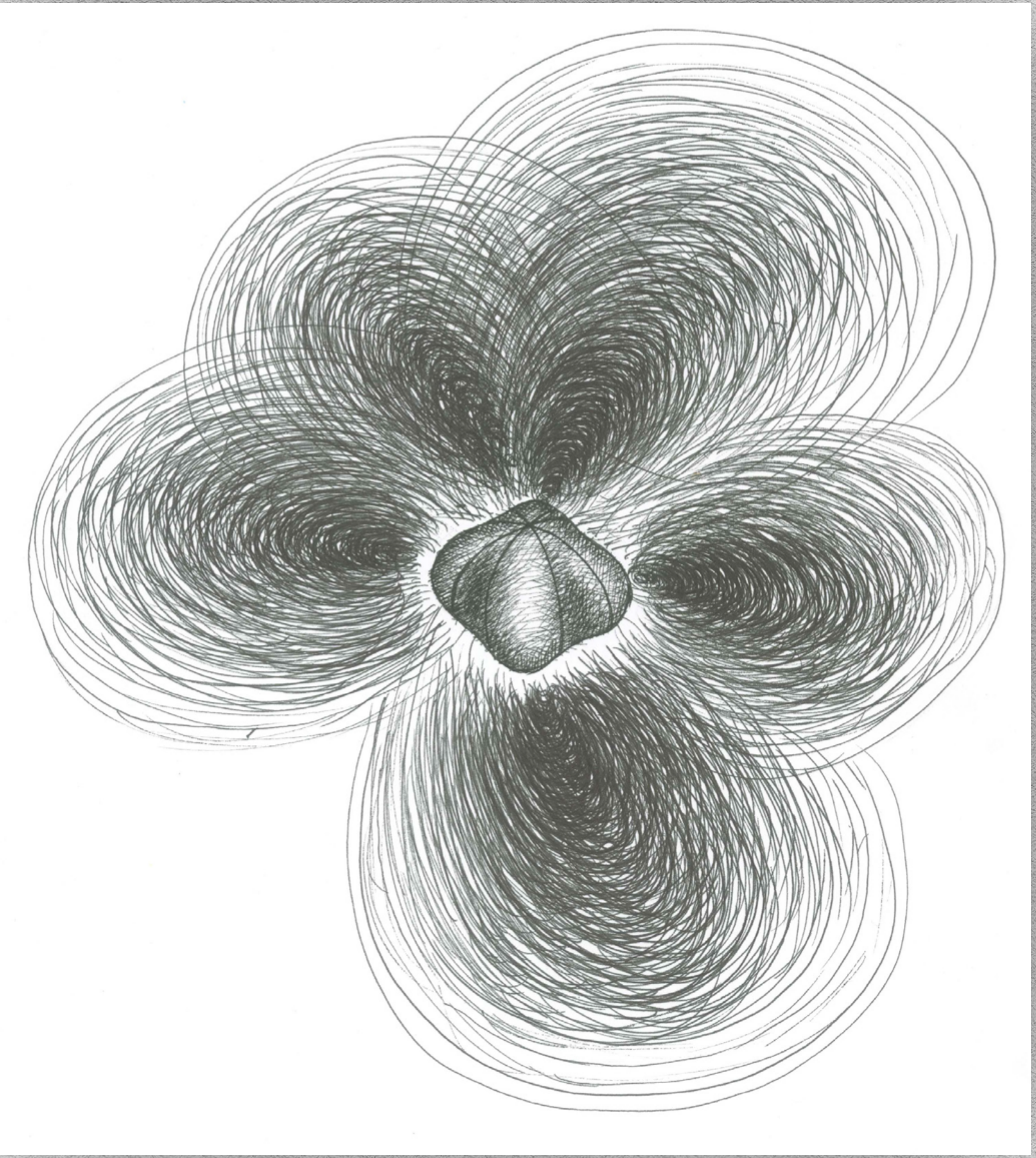
Neutron star oscillations: dynamics & gravitational waves

NewCompStar School 2016
Coimbra, Portugal



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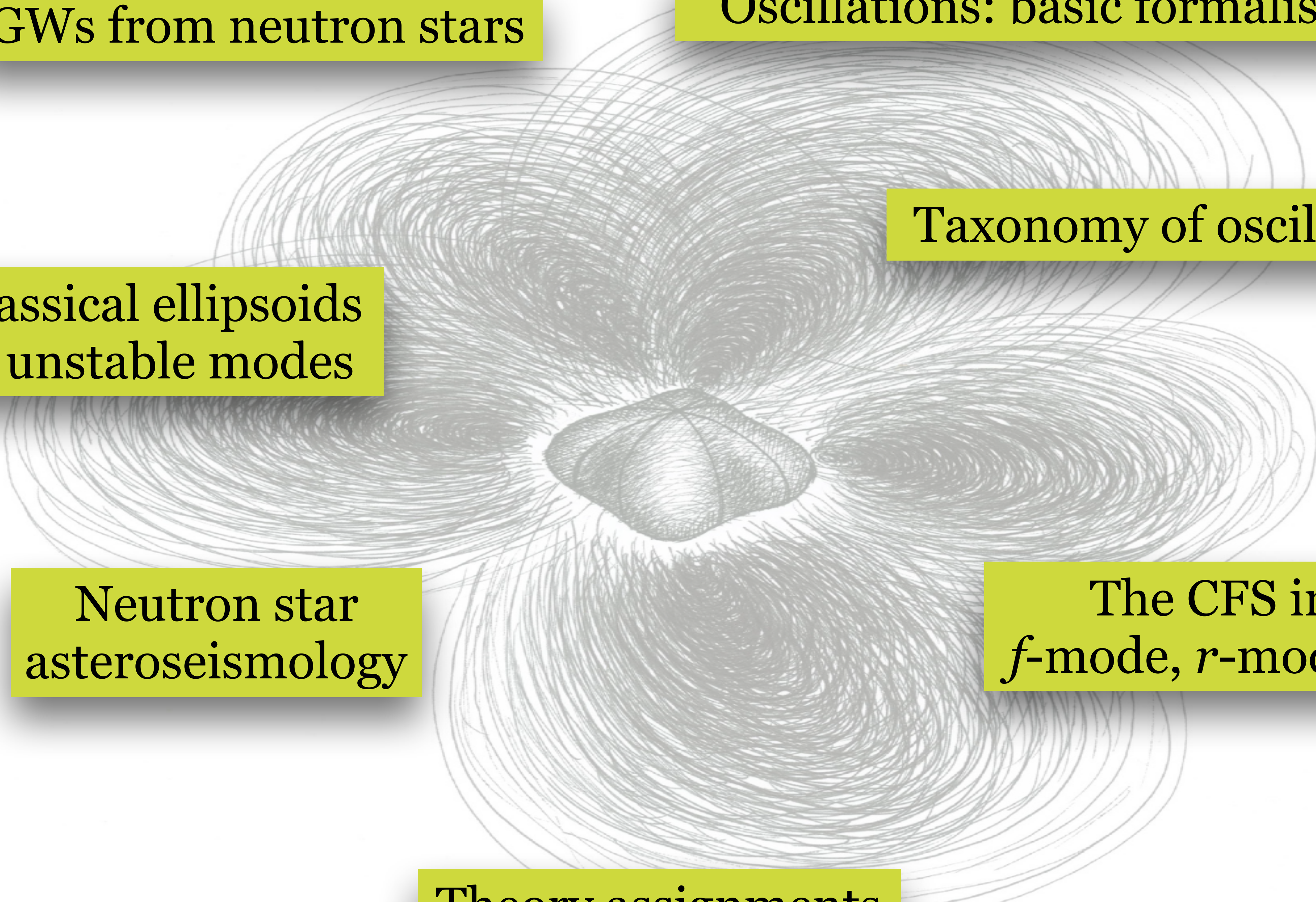


GWs from neutron stars

Oscillations: basic formalism

Taxonomy of oscillations modes

Classical ellipsoids & unstable modes

Neutron star asteroseismology

The CFS instability: f-mode, r-mode & w-mode

Theory assignments

A note on *Notation*

Abbreviations:

NS = neutron star

BH = black hole

GW = gravitational waves

EM = electromagnetic waves

LMXB = low mass x-ray binary

SGR = soft gamma repeater

MSP = millisecond radio pulsar

EOS = equation of state

sGRB = short gamma-ray burst

Basic parameters:

$M =$ stellar (gravitating) mass $\quad M_{1.4} = \dfrac{M}{1.4 M_\odot}$

$R =$ stellar radius $\quad R_6 = \dfrac{R}{10^6\ \mathrm{cm}}$

$\rho =$ density

$T =$ stellar core temperature

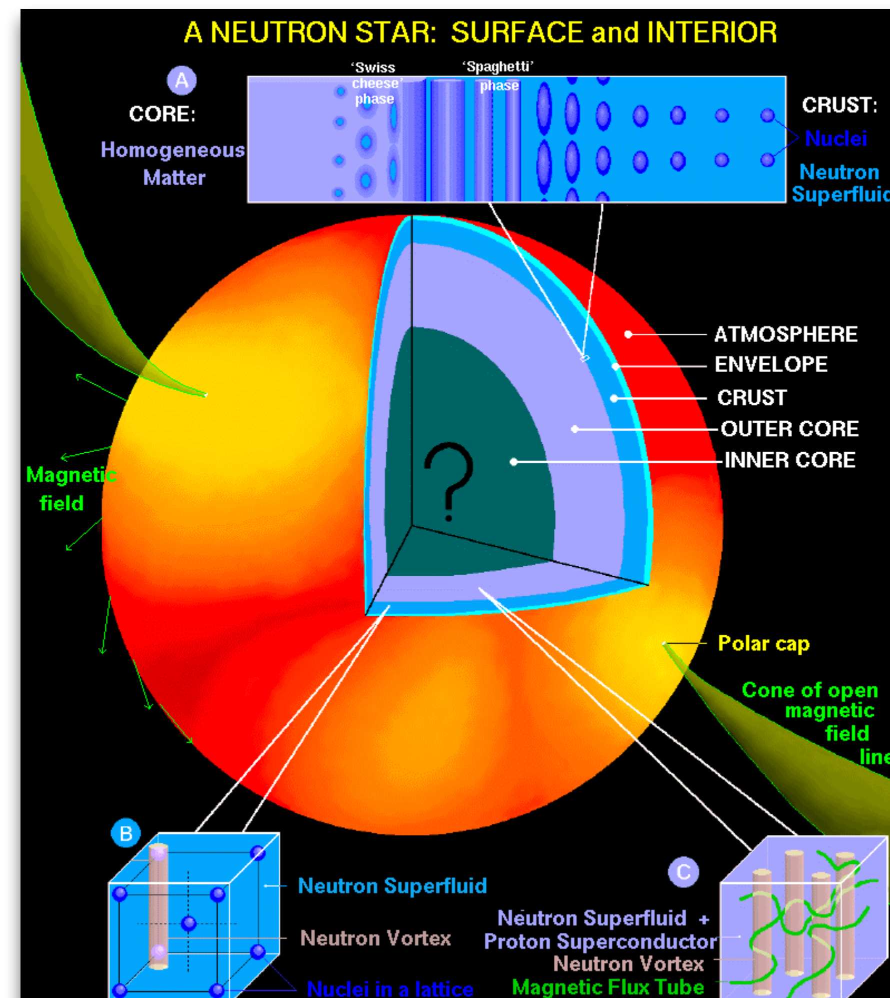
$\omega =$ mode's angular frequency

$\Omega =$ rotational angular frequency

$f_{\rm spin} = \dfrac{1}{P} =$ rotational frequency & period

A cosmic laboratory of matter & gravity

- Supranuclear equation of state (hyperons, quarks)
- Relativistic gravity
- Rotation (oblateness, various instabilities)
- Magnetic fields (configuration, stability)
- Elastic crust (fractures)
- Superfluids/superconductors (multi-fluids, vortices, fluxtubes)
- Viscosity (mode damping)
- Temperature profiles (exotic cooling mechanisms)



[figure: D. Page]

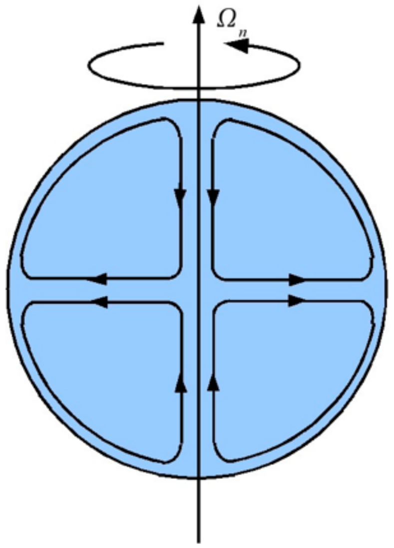
Neutron stars as GW sources (I)

“Burst” emission

Binary neutron star mergers
(our safest bet for detection)

Magnetar flares
(likely too weak)

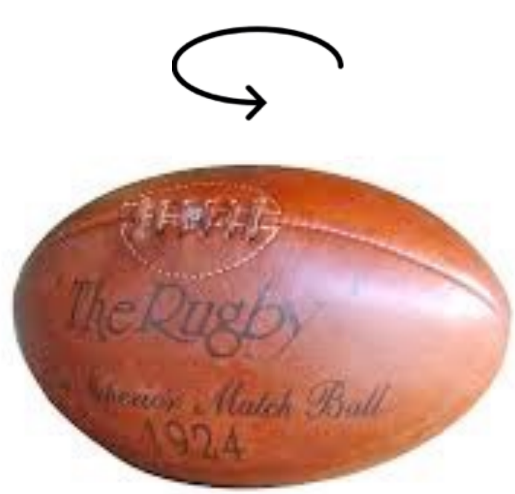
Pulsar glitches
(likely too weak)



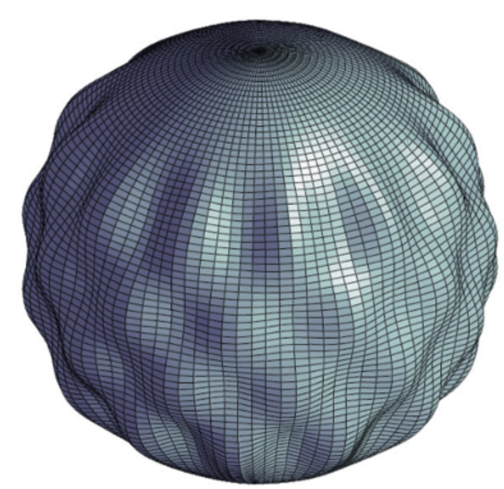
Neutron stars as GW sources (II)

Continuous emission

Non-axisymmetric mass quadrupole ("mountains")



Fluid part (oscillations)



Taxonomy of NS oscillation modes (I)

- **Pressure (*p*) modes:** driven by pressure.
- **Fundamental (*f*) mode:** (aka “Kelvin mode”) the first (nodeless) *p*-mode.
- **Gravity (*g*) modes:** driven by buoyancy (thermal/composition gradients).
- **Inertial (*i*) modes:** driven by rotation (Coriolis force).
- **Magnetic (Alfven) modes:** driven by the magnetic force.
- **Spacetime (*w*) modes:** akin to BH QNMs, need dynamical spacetime (non-existent in Newtonian gravity)

Taxonomy of NS oscillation modes (II)

More physics in stellar model $\Rightarrow$ richer mode spectrum

- **Shear (*s*,*t*) modes:** driven by elastic forces in the crust.

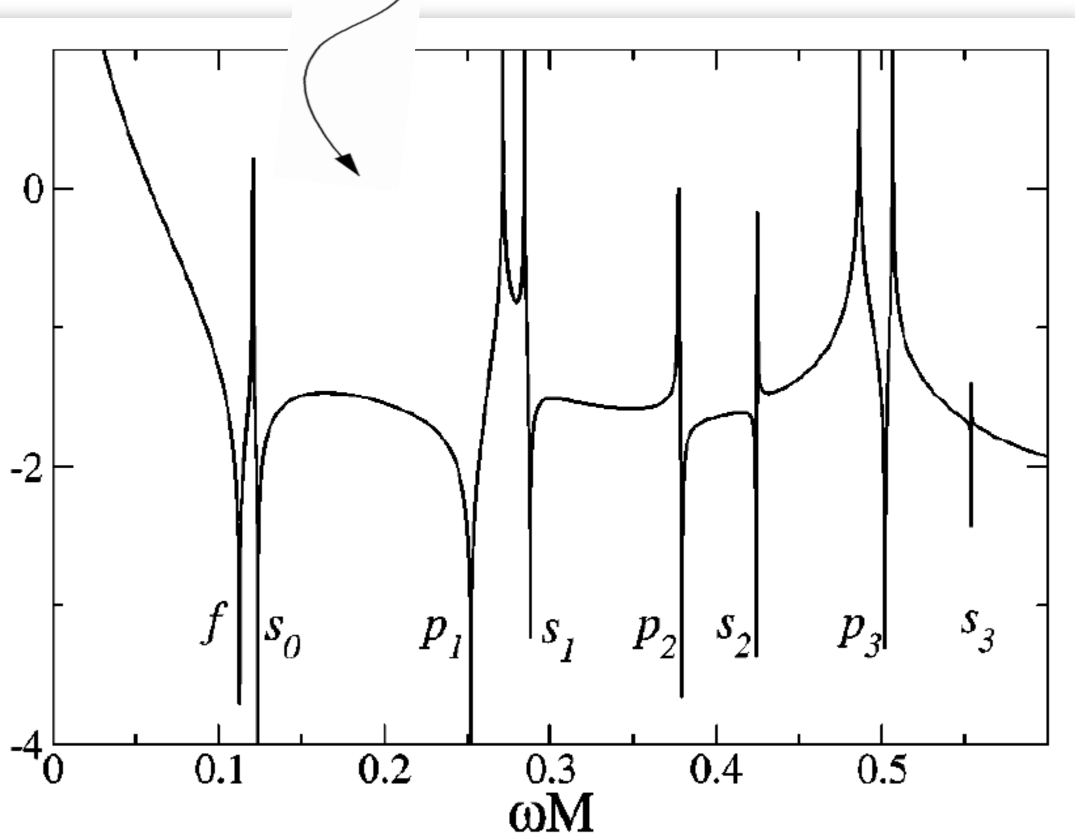
- **Superfluidity:** the system becomes a multi-fluid (i.e. relative motion of one fluid with respect to the others). Modes are "doubled", due to the "co-moving" and "counter-moving" degrees of freedom.

- **Tkachenko modes:** driven by tension of superfluid vortex array (never computed for NS, except from local plane waves).

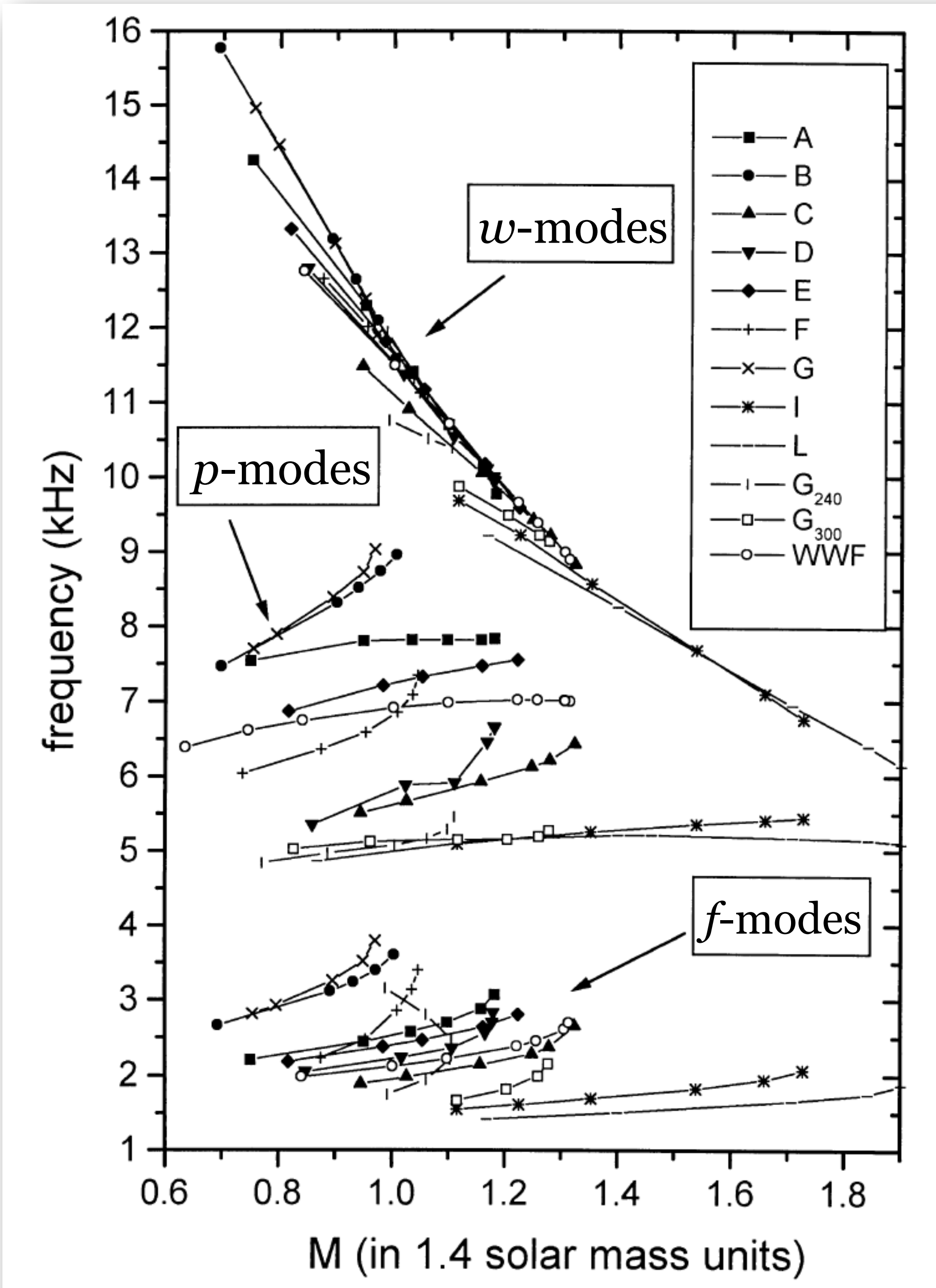
Taxonomy of NS oscillation modes (III)

Typical spectra of *non-rotating* NSs without stratification

A typical spectrum "doubling" in a non-rotating *superfluid* NS



[Andersson et al. 2002]



[Kokkotas et al. 2001]

NS modes: geometry

- The velocity perturbation associated with a mode can be decomposed in a standard way in radial and angular parts:

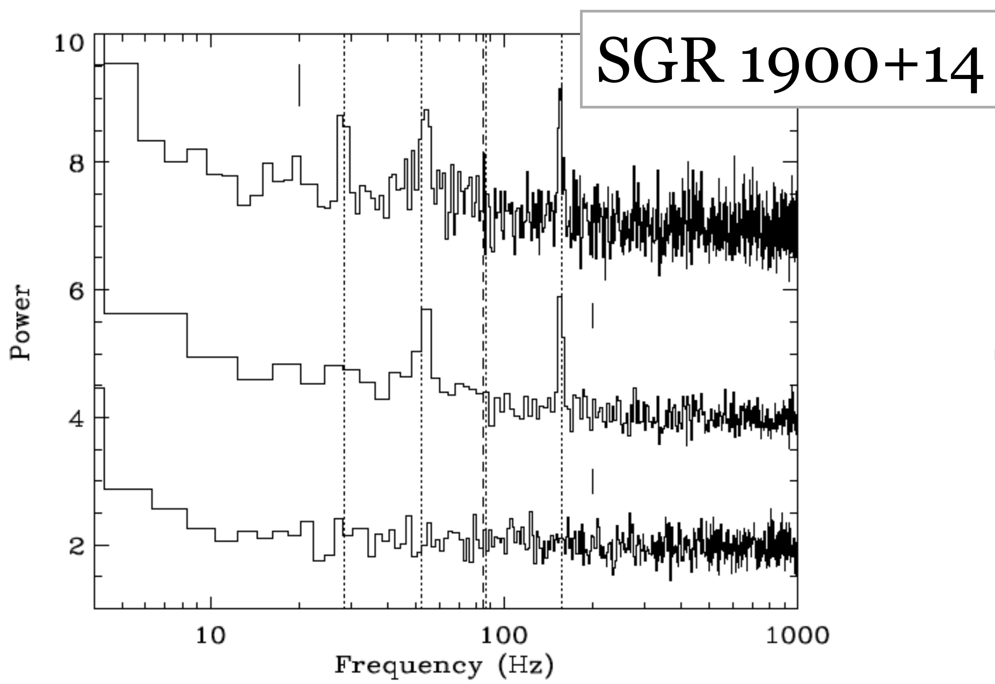
$$\delta \mathbf{v}(\mathbf{x},t) = \sum_{\ell,m} \left[\underbrace{W_\ell\, \hat{\mathbf{r}} + V_\ell \boldsymbol{\nabla} Y_\ell^m}_{\textit{polar}\text{ part = parity}\,(-1)^\ell} + \underbrace{U_\ell\, (\hat{\mathbf{r}} \times \boldsymbol{\nabla} Y_\ell^m)}_{\textit{axial}\text{ part = parity}\,(-1)^{\ell+1}} \right] e^{i\omega t}$$

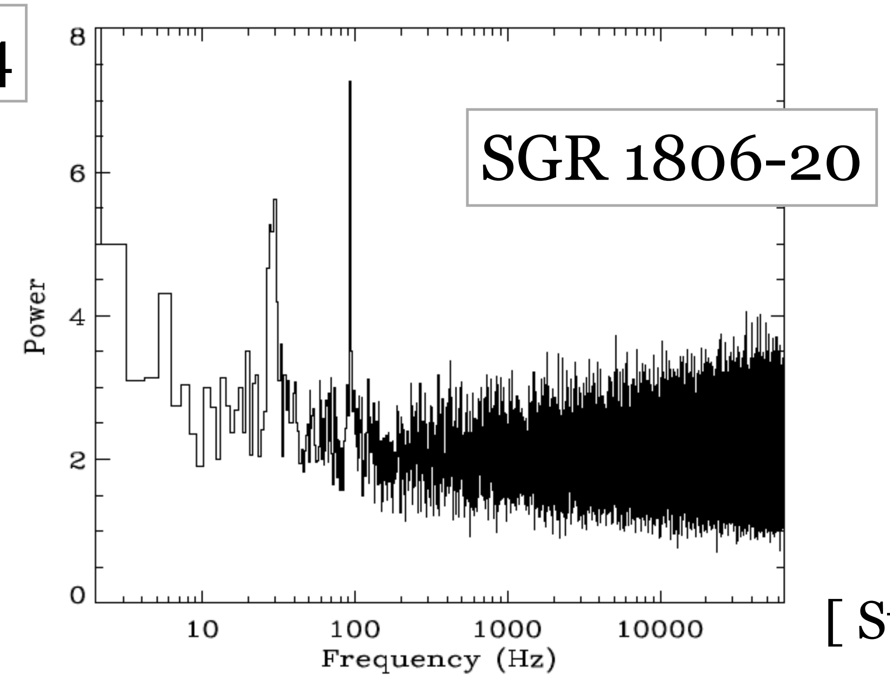
radial eigenfunctions: $W_\ell(r), V_\ell(r), U_\ell(r)$

- In *spherical stars* (i.e. up to $\mathcal{O}(\Omega)$), axial and polar sectors remain *decoupled*.
- *Purely polar*: *f*-mode, *p*-modes, *g*-modes, ...
- *Purely axial*: *r*-modes, *t*-crust modes, ... $\Rightarrow \boldsymbol{\nabla} \cdot \delta \mathbf{v} = 0$ & flow “horizontal”
- Coupling: with rotation ($\mathcal{O}(\Omega^2)$ and higher), B-field, ...
- Similar decomposition in GR stars

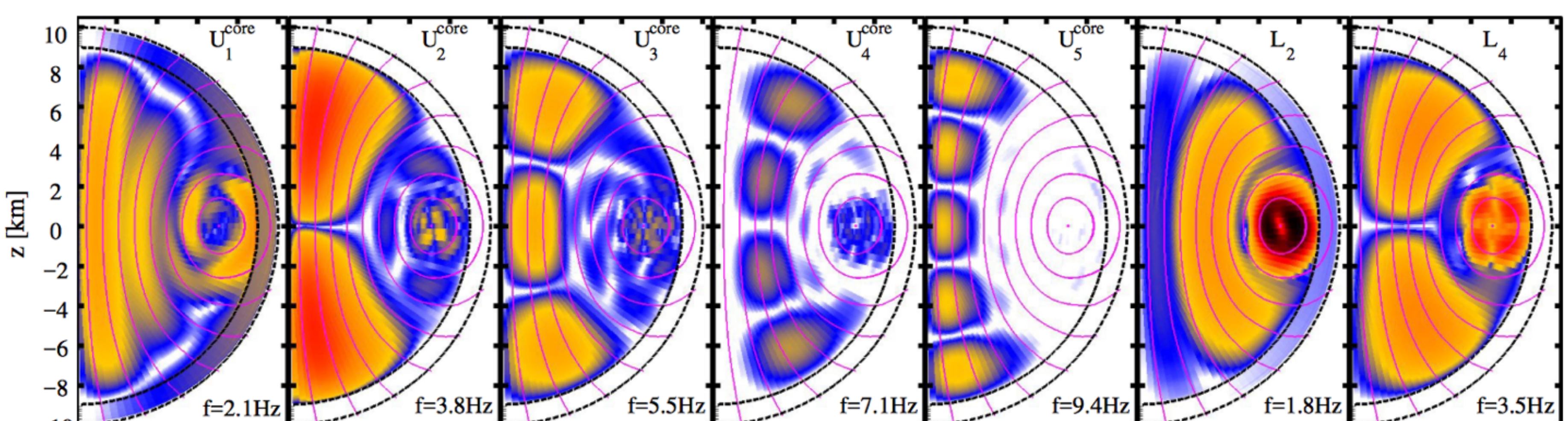
NS modes observed: magnetar flares

- Quasi-periodic oscillations in the x-ray light curve of giant flares in SGRs.
- These are believed to be global magnetic/magneto-elastic modes.





[Strohmayer & Watts 2005, 2006]



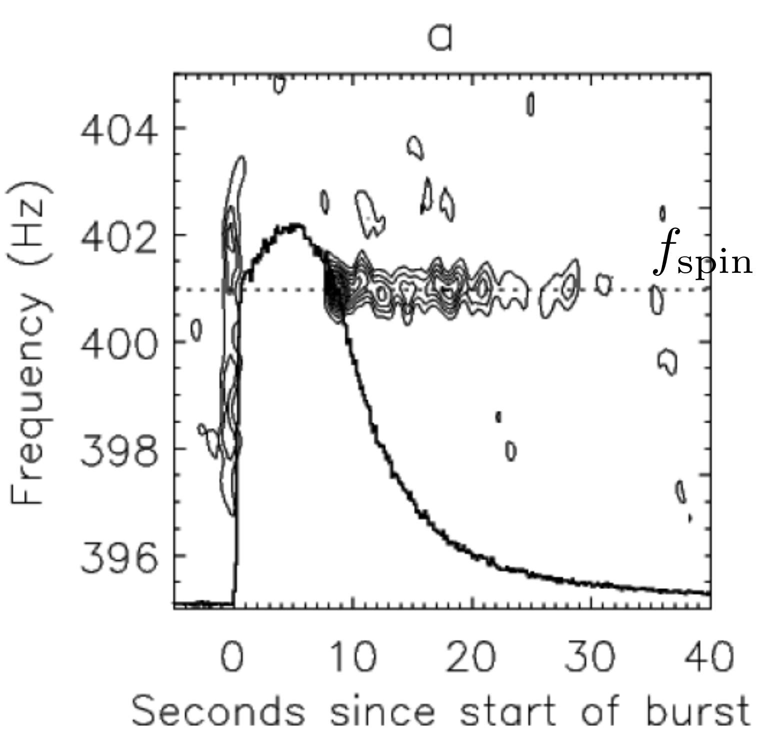
[Gabler et al. 2016]

NS modes observed: bursting LMXBs

- NSs in LMXBs frequently undergo x-ray burst whose light curves are oscillatory. Two main models:

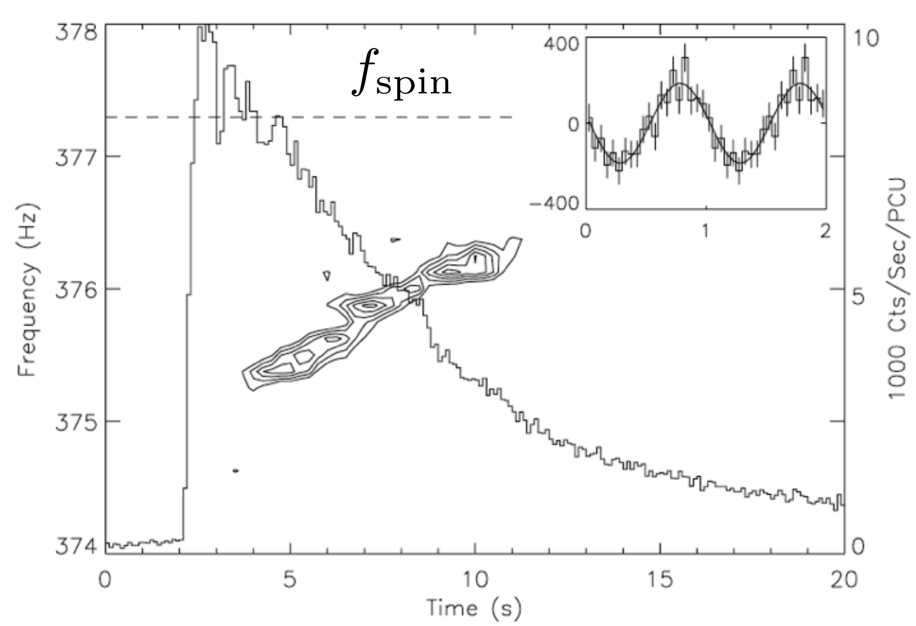
- Surface modes (*r*-modes, *g*-modes ...) in fluid ocean, excited by infalling matter and burning.

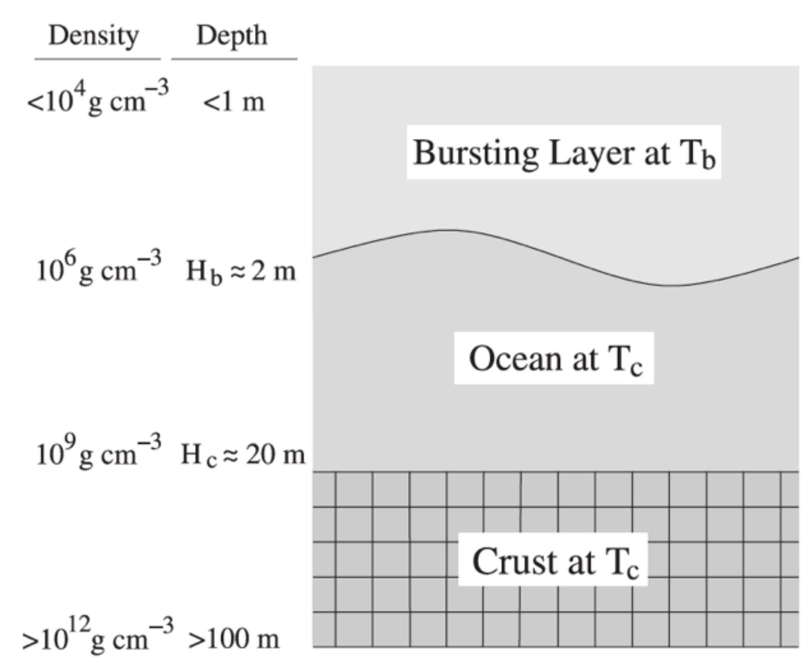
- Surface "hot spot" emission modulated by rotation.



[Chakrabarty et al. 2003]

[Watts et al. 2009]





[Piro & Bildsten 2005]

NS modes: basic formalism (I)

- Linearised equations, written in the stellar *rotating* frame.
- Mass continuity equation: $\partial_t \delta\rho + \boldsymbol{\nabla} \cdot (\rho \delta \mathbf{v}) = 0$
- Poisson equation: $\nabla^2 \delta\Phi = 4\pi G \delta\rho$
- Euler (or Navier-Stokes) equation:

$$\partial_t \delta \mathbf{v} + 2\boldsymbol{\Omega} \times \delta \mathbf{v} + \boldsymbol{\nabla}\left(\frac{\delta p}{\rho} + \delta\Phi_{\rm eff}\right) = \frac{1}{\rho}\left(\underbrace{\mathbf{F}_{\rm sv} + \mathbf{F}_{\rm bv}}_{\text{shear \& bulk viscous forces}} + \overbrace{\mathbf{F}_{\rm GR}}^{\text{GW radiation reaction force}}\right) + \underbrace{\frac{1}{\rho}\mathbf{F}_{\rm mag}}_{\text{magnetic force}} + \{\ldots\}$$

- A *barotropic* EOS *p*= *p*(*ρ*) was assumed (realistic NSs are *not* barotropes).
- Superfluid NSs require a *multi-fluid* formalism, instead of a single-fluid one.
- In the presence of a magnetic field, the Maxwell equations have to be added.

NS modes: basic formalism (II)

- The mode's total energy $E_{\rm mode}$ is conserved in the absence of dissipation.
- In the presence of dissipation $E_{\rm mode}$ is not conserved and the mode's frequency ω becomes complex-valued.
- The Navier-Stokes equation leads to:

$$\begin{cases} \dot{E}_{\rm mode} = -\dfrac{2E_{\rm mode}}{\tau} \\ \dot{E}_{\rm mode} = \dot{E}_{\rm sv} + \dot{E}_{\rm bv} + \dot{E}_{\rm GRR} \end{cases}$$

where

$$\mathrm{Im}(\omega) = \frac{1}{\tau}$$

shear viscosity damping rate:

$$\dot{E}_{\rm sv} = -2\int dV \eta\, \delta\sigma^{ij}\delta\bar{\sigma}_{ij}$$

$$\delta\sigma^{ij} = \frac{1}{2}\left(\nabla^i\delta v^j + \nabla^j\delta v^i - \frac{2}{3}g^{ij}\nabla_k\delta v^k\right)$$

$$\frac{1}{\tau_{\rm sv}} = \frac{\dot{E}_{\rm sv}}{E_{\rm mode}}$$

bulk viscosity damping rate:

$$\dot{E}_{\rm bv} = -\int dV \zeta\, |\delta\sigma|^2$$

$$\delta\sigma = \nabla_j\delta v^j$$

$$\frac{1}{\tau_{\rm bv}} = \frac{\dot{E}_{\rm bv}}{E_{\rm mode}}$$

Calculating mode damping: basic strategy

Solve directly the Navier-Stokes equation

daunting task due to complexity of dissipative forces

mind causality of viscosity in GR!

realistic scenario: weak dissipation

$\mathrm{Re}(\omega) \gg |\mathrm{Im}(\omega)|$

Solve the non-dissipative Euler equation

use inviscid mode eigenfunctions & frequencies in $\dot{E}_{\rm sv}, \dot{E}_{\rm bv}, \dot{E}_{\rm GRR}$ to obtain approximate viscous and GW timescales.

NS modes: GR formalism

- The formalism is considerably more complicated in GR:

$$\delta G^{\mu\nu} = 8\pi \delta T^{\mu\nu} \quad \delta(\nabla_\nu T^{\mu\nu}) = 0$$

metric = $g_{\mu\nu} + \delta g_{\mu\nu}$

Cowling approximation: $\delta g_{\mu\nu} = 0$

$\Rightarrow$ switches off GWs but also "contaminates" mode

- Example: the symbolic form of polar perturbation equations in a spherical background star

$$-\frac{1}{c^2}\frac{\partial^2 S}{\partial^2 t} + \frac{\partial^2 S}{\partial^2 r_*} + L_1(S, F, \ell) = 0$$

$$-\frac{1}{c^2}\frac{\partial^2 F}{\partial^2 t} + \frac{\partial^2 F}{\partial^2 r_*} + L_2(S, F, H, \ell) = 0$$

$$-\frac{1}{(c_s)^2}\frac{\partial^2 H}{\partial^2 t} + \frac{\partial^2 H}{\partial^2 r_*} + L_3(H, H', S, S', F, F', \ell) = 0$$

+ constraint:

$$\frac{\partial^2 F}{\partial^2 r_*} + L_4(F, F', S, S', H, \ell) = 0$$

f-mode: back of the envelope

- The "minimal" (Newtonian) stellar model supporting f-modes:

uniform density	incompressible flow	Cowling approximation
$\delta\rho = 0$	$\nabla \cdot \delta\mathbf{v} = 0$	$\delta\Phi = 0$

Euler: $$\partial_t \delta\mathbf{v} + \frac{1}{\rho}\nabla\delta p = 0 \Rightarrow \begin{cases} \delta\mathbf{v} = \partial_t \boldsymbol{\xi} = \nabla\chi & \text{“potential flow”} \\ \nabla^2\delta p = 0 \quad \& \quad \nabla^2\chi = 0 \end{cases}$$

$$\begin{cases} \chi = \chi_\ell(r) Y_\ell^m e^{i\omega t} \\ \delta p = \delta p_\ell(r) Y_\ell^m e^{i\omega t} \end{cases} \Rightarrow \begin{cases} \chi_\ell = \alpha_\ell r^\ell \\ \delta p_\ell = \beta_\ell r^\ell \end{cases}$$

Euler: $$i\omega\alpha_\ell + \frac{\beta_\ell}{\rho} = 0$$

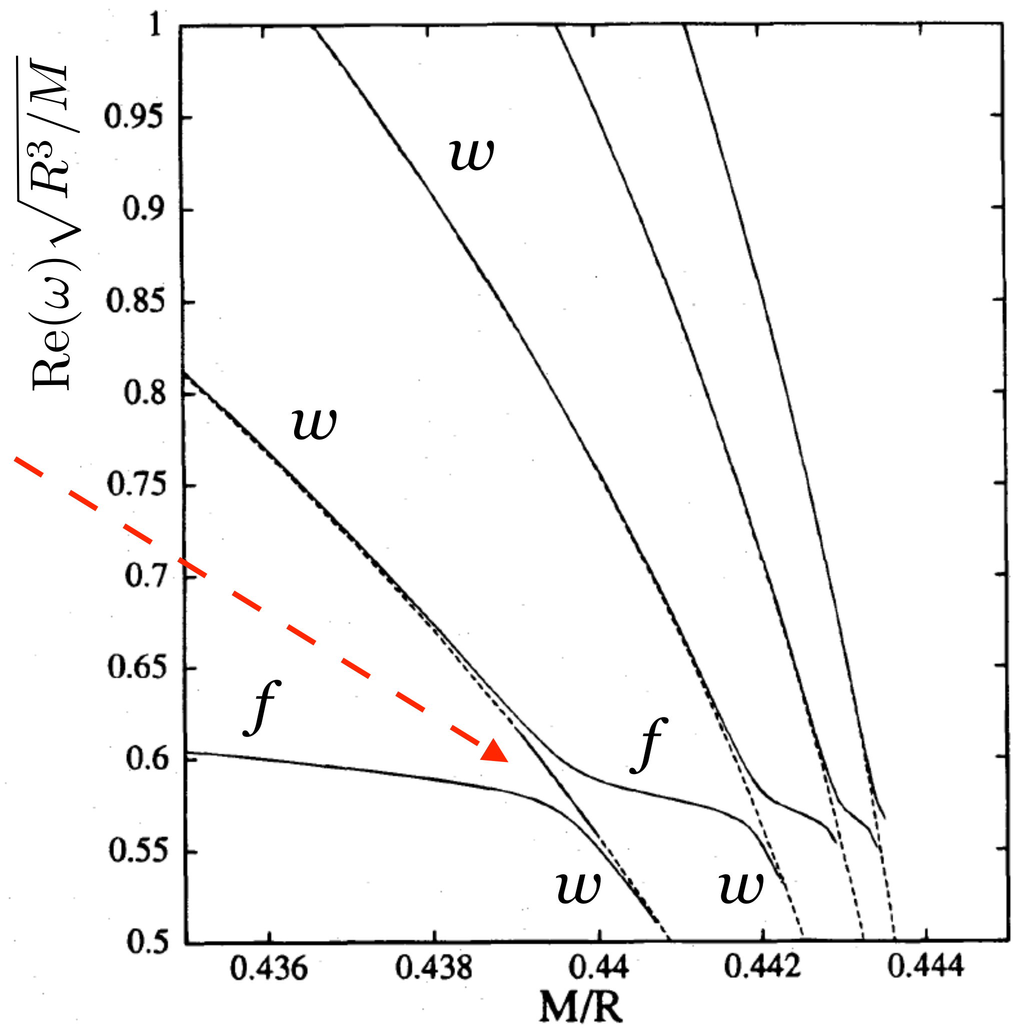
background pressure gradient

Surface boundary condition (r=R): $$\Delta p = \delta p + \xi^r \partial_r p = 0$$

$$\Rightarrow \quad \beta_\ell = \frac{4\pi G\rho^2\ell}{3i\omega}\alpha_\ell \quad \Rightarrow \quad \omega^2 = \frac{4\pi\ell}{3}G\rho \quad \rightsquigarrow \quad \omega \sim \sqrt{G\rho}$$

A simple GR f-mode calculation

- Ultracompact, uniform fluid ball (i.e. the Schwarzschild solution).
- The system can only support w-modes and the fluid f-mode (only the latter in Newtonian gravity).
- The figure provides a beautiful example of mode *avoided crossings*.
- At each crossing the two modes "transmute" by exchanging properties.
- In this particular example, the avoided crossings "produce" the f-mode.



[Andersson et al. 1996]

GW asteroseismology

- *Key idea of asteroseismology:*

  parametrise mode frequencies & decay rates (due to GW emission) in terms of the bulk stellar parameter: $\{M, R, \Omega\}$

  Once an oscillation is observed, use the parametrisation to infer the stellar parameters.

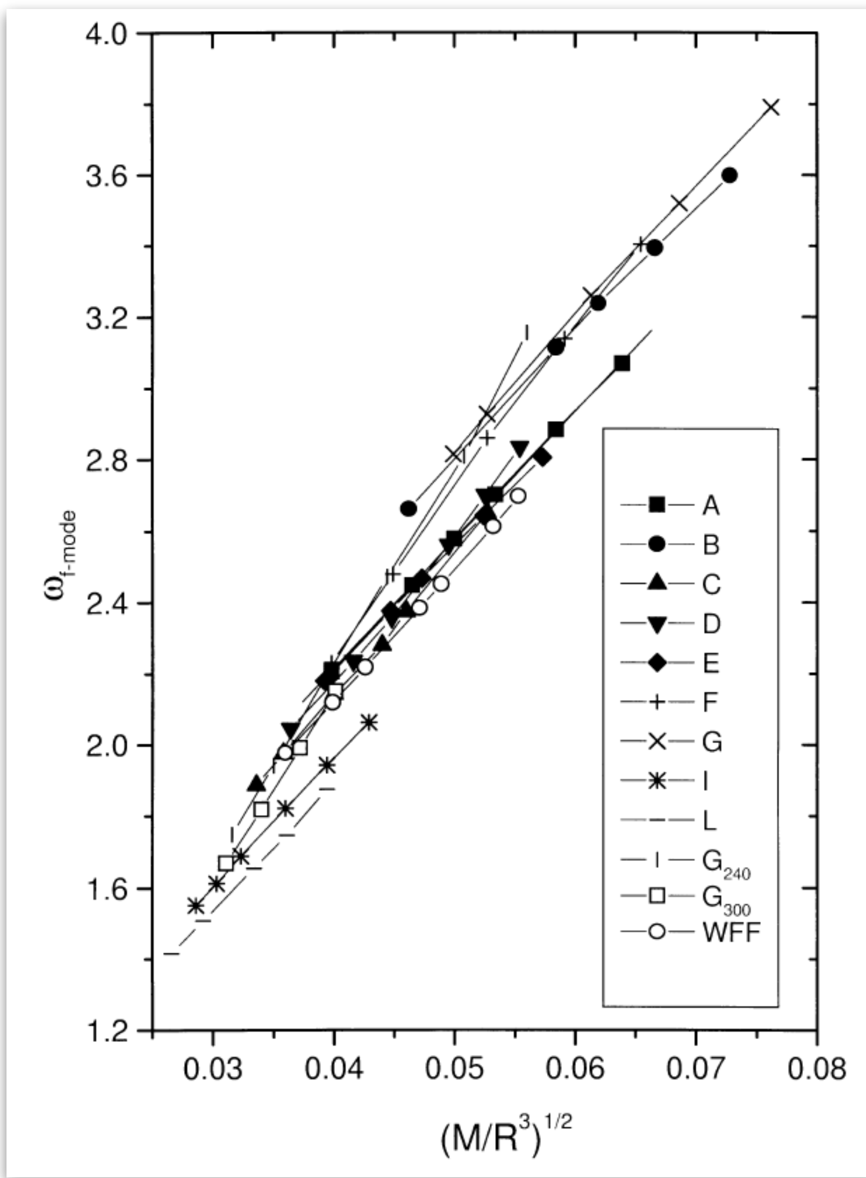
- A clever parametrisation can lead to "universal" (i.e. quasi EOS-independent) relations.

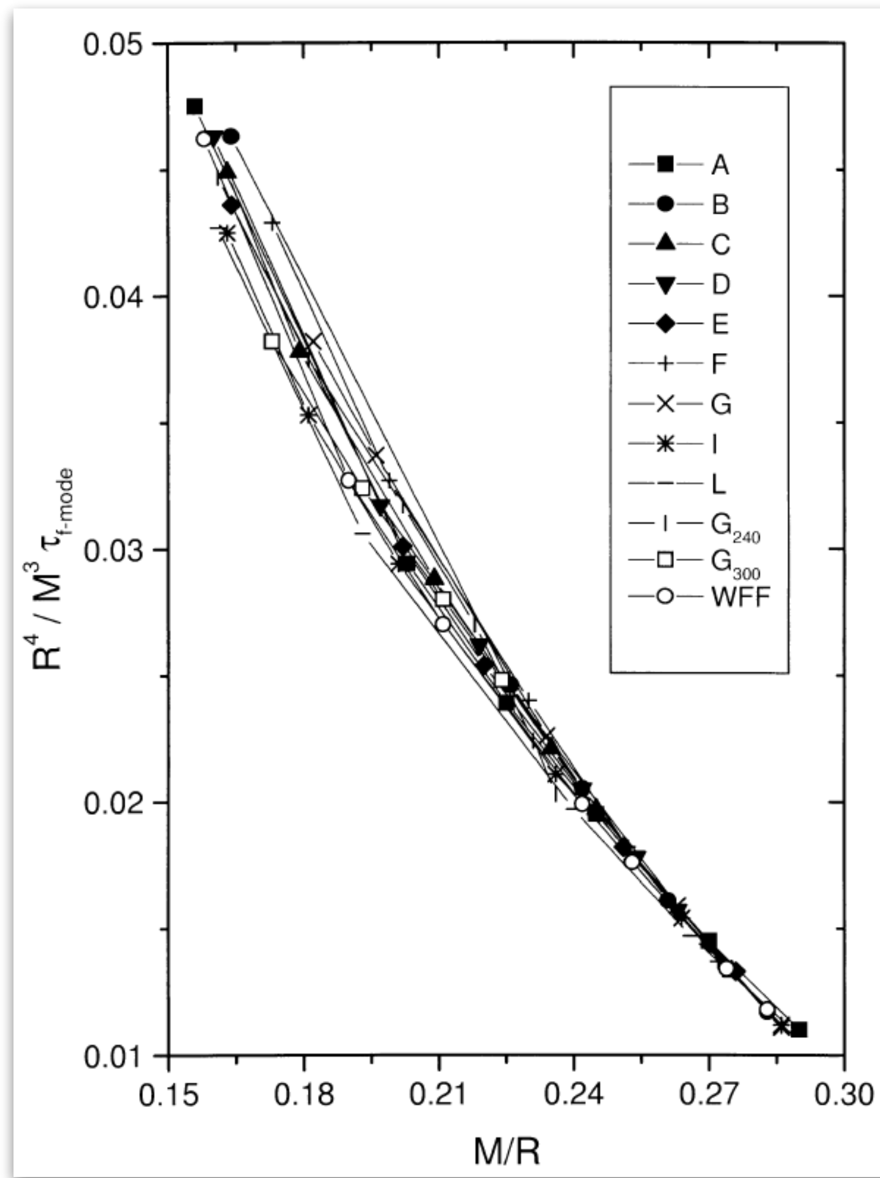
- As an example, we consider *f*-mode asteroseismology.

f-mode asteroseismology: *no* rotation

- Fitting formulae for mode frequency and GW decay time.

$$\omega_f(\mathrm{kHz}) \approx 0.78 + 1.635 \left(\frac{M_{1.4}}{R_6^3} \right)^{1/2} \qquad \tau_f(s) \approx \frac{R_6^4}{M_{1.4}^3} \left[22.85 - 14.65 \frac{M_{1.4}}{R_6} \right]^{-1}$$





[Andersson & Kokkotas 1998]

f-mode asteroseismology: *with* rotation

- For rotating NSs we need to consider the prograde (stable) and retrograde (potentially unstable) f-modes.

$$\frac{\omega_r^u}{\omega_0} = 1 + 0.402\frac{\Omega}{\Omega_{\rm K}} - 0.406\left(\frac{\Omega}{\Omega_{\rm K}}\right)^2$$

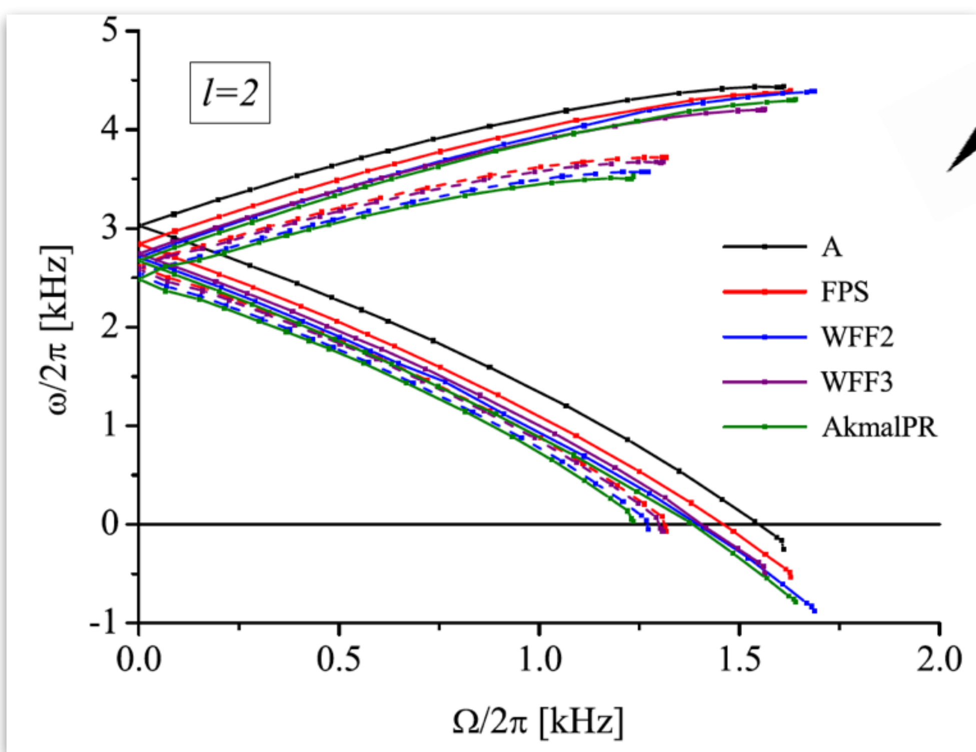
$$\frac{\omega_r^s}{\omega_0} = 1 - 0.235\frac{\Omega}{\Omega_{\rm K}} - 0.358\left(\frac{\Omega}{\Omega_{\rm K}}\right)^2$$

$\Omega_{\rm K}$ = Kepler frequency

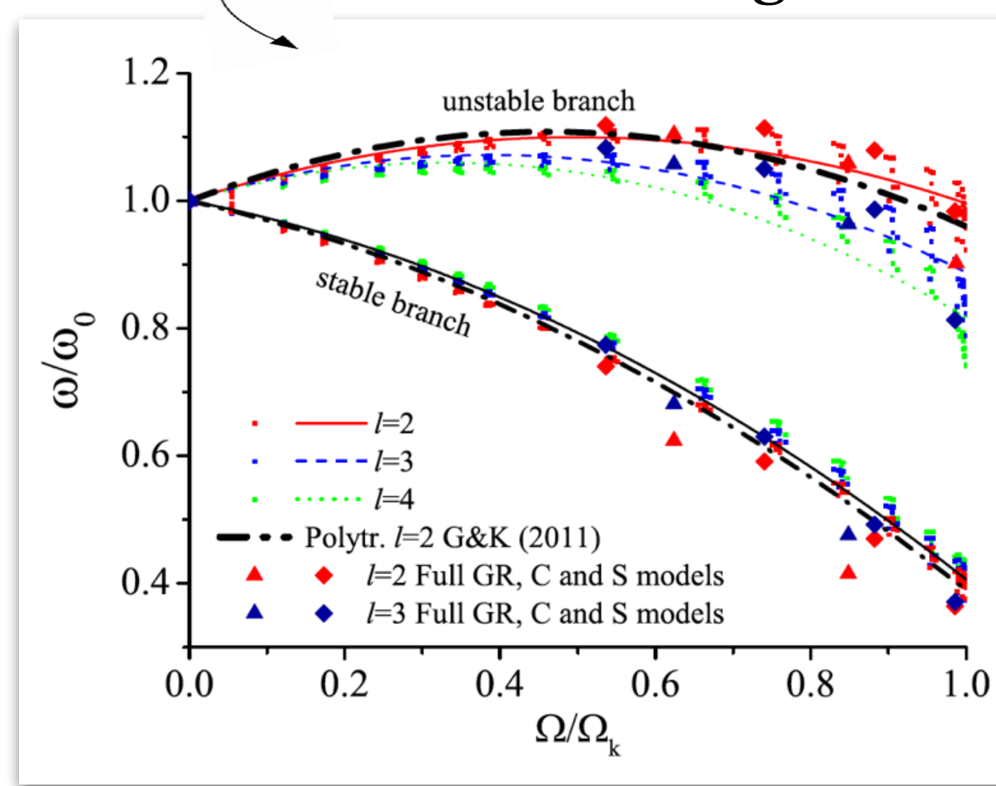
ω_0 = non-rotating f-mode frequency

$$\omega_i = \omega_r - m\Omega$$

inertial frame



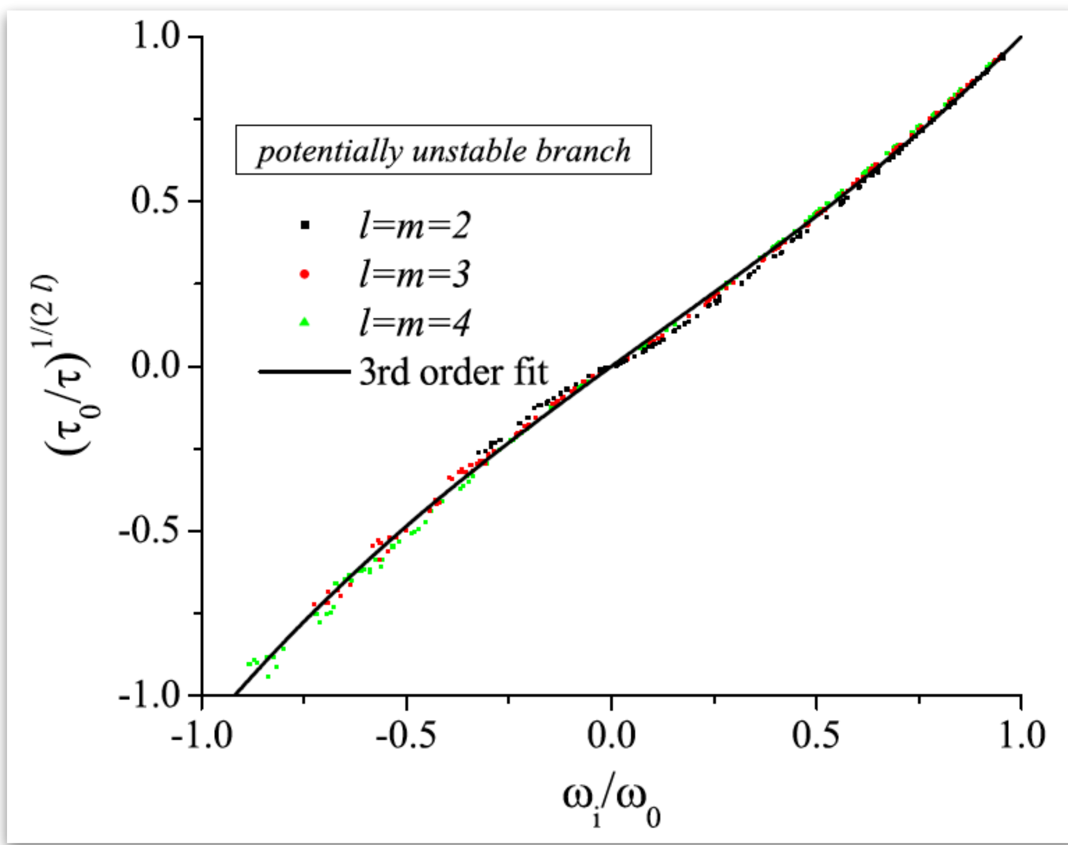
rotating frame

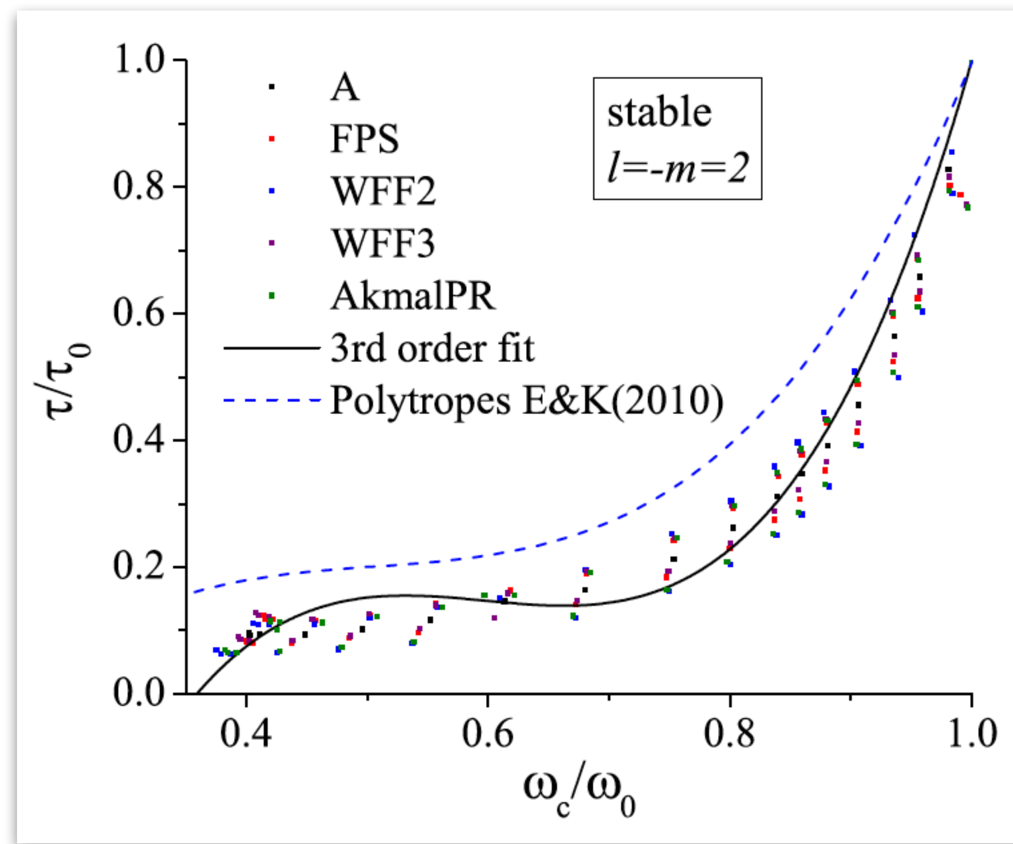


[Doneva & Kokkotas 2013]

f-mode asteroseismology: *with* rotation

- Polynomial fitting formulae exist for the *f*-mode's GW damping timescales.





[Doneva & Kokkotas 2013]

Unstable modes & Ellipsoids (I)

- A 300 years-old question:
  what is the equilibrium shape of a rotating self-gravitating fluid body?
- We consider homogenous & incompressible bodies.
- *Maclaurin* (1742): body is oblate and *biaxial*, the angular frequency Ω and ellipticity e are related as:

$$\Omega^2 = 2\pi G\rho \left[\frac{(1-e^2)^{1/2}}{e^3}(3-2e^2)\sin^{-1} e - \frac{3(1-e^2)}{e^2} \right] \qquad e = \sqrt{1-\left(\frac{a_3}{a_1}\right)^2}$$

- *Jacobi* (1834): equilibrium shape can be *triaxial* ellipsoidal.

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} = 1$$

- The fluid velocity for both configurations is a linear function of coordinates:

$$\mathbf{v} = \Omega(\,-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}\,)$$

Unstable modes & Ellipsoids (II)

- The Maclaurin sequence *bifurcates* at:

$$e \approx 0.813 \qquad \beta \approx 0.14$$

where

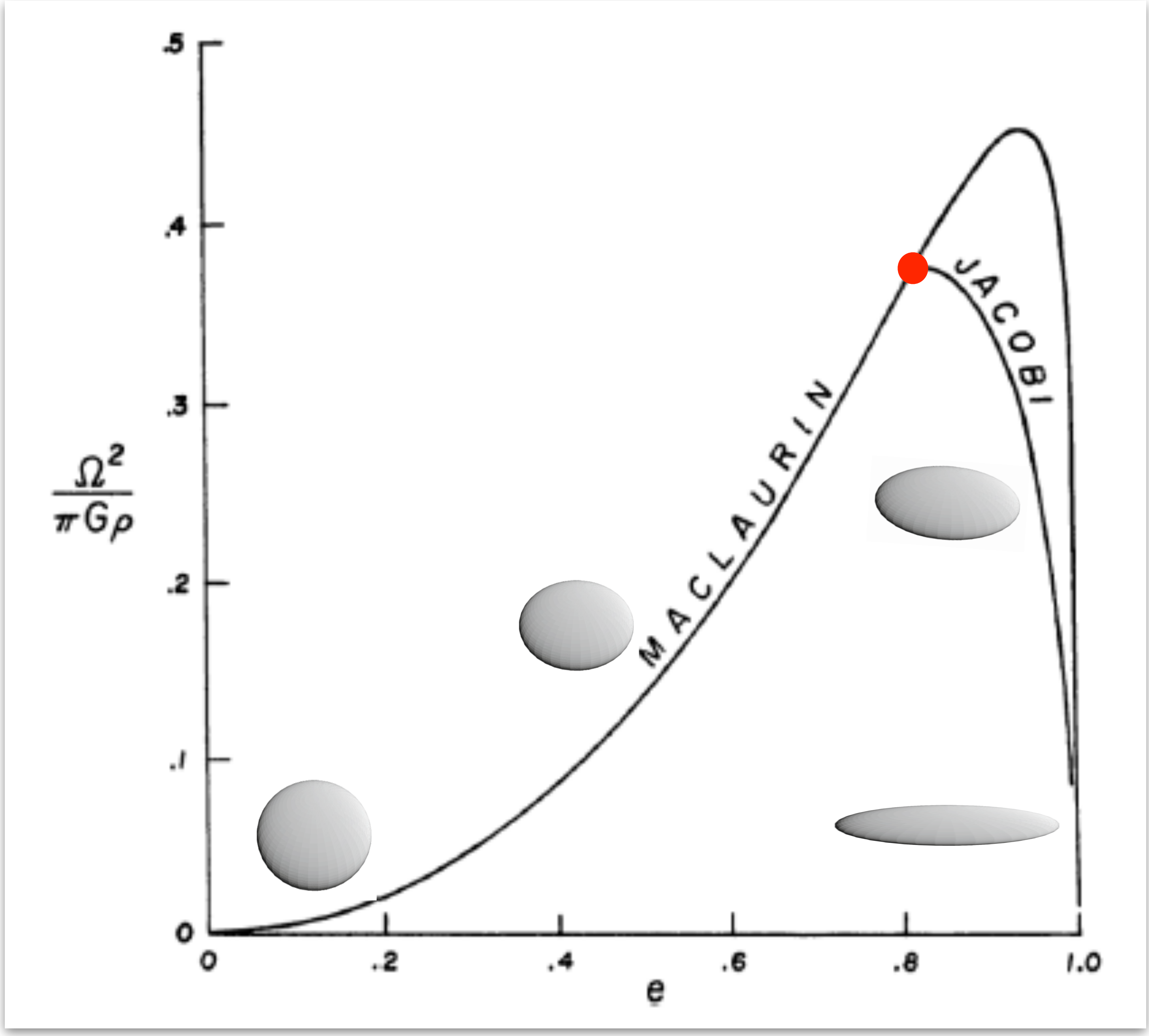
$$\beta = \frac{T}{W} = \frac{\text{kinetic energy}}{\text{grav. potential energy}}$$

Jacobi sequence ends in a "cigar":

$$e \to 1 \;\Rightarrow\; \frac{a_3}{a_1} \to 0, \quad \frac{a_3}{a_2} \to 1$$

Maclaurin sequence ends in a "disk":

$$e \to 1 \;\Rightarrow\; \frac{a_3}{a_1} \to 0, \quad J \to \infty$$



Unstable modes & Ellipsoids (III)

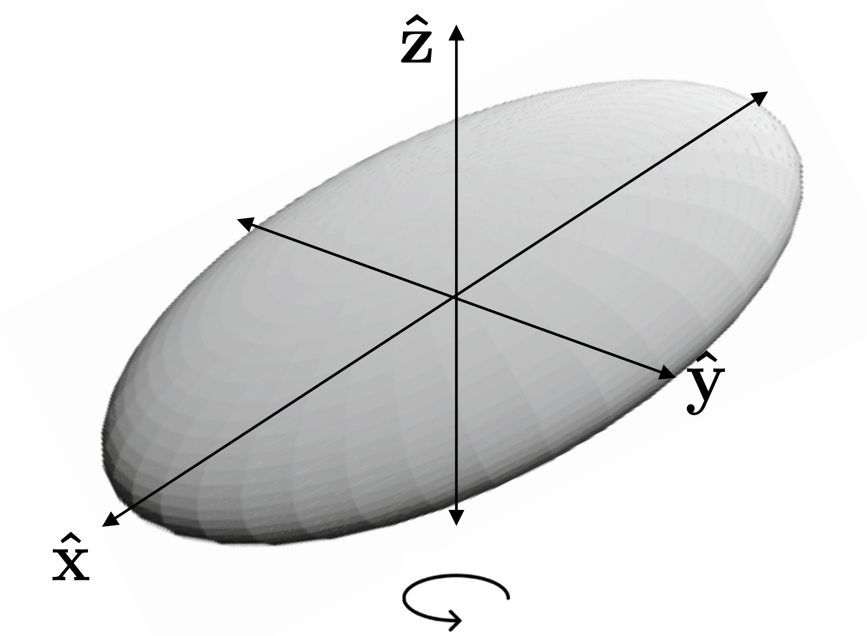
- *Dirichlet-Dedekind* (1861): a new class of triaxial ellipsoids, with zero rigid body rotation and non-zero uniform *vorticity*

$$\Omega = 0 \qquad \boldsymbol{\zeta} = \boldsymbol{\nabla} \times \mathbf{v} \neq 0 \qquad \text{velocity:} \quad \mathbf{v} = \frac{\zeta}{a_1^2 + a_2^2}\left(-a_2^2\, y\, \hat{\mathbf{x}} + a_1^2\, x\, \hat{\mathbf{y}} \right)$$

- The previous solutions are special cases of the general *Riemann* family of ellipsoids.

- *S-type* ellipsoids: $\left\{ \begin{array}{l} \boldsymbol{\Omega} \parallel \boldsymbol{\zeta} \ \text{ along a principal axis of the ellipsoid,} \quad \dfrac{\zeta}{\Omega} = \text{const.} \\ \text{linear fluid flow } v^i = A^{ij} x_j \end{array} \right.$

$$\parallel \hat{\mathbf{z}} \left\{ \begin{array}{l} \boldsymbol{\Omega} = (0, 0, \Omega) \\ \boldsymbol{\zeta} = (0, 0, \zeta) \end{array} \right. \qquad \frac{\zeta}{\Omega} = \left\{ \begin{array}{ll} 0 & \text{Jacobi} \\ \infty & \text{Dedekind} \end{array} \right.$$



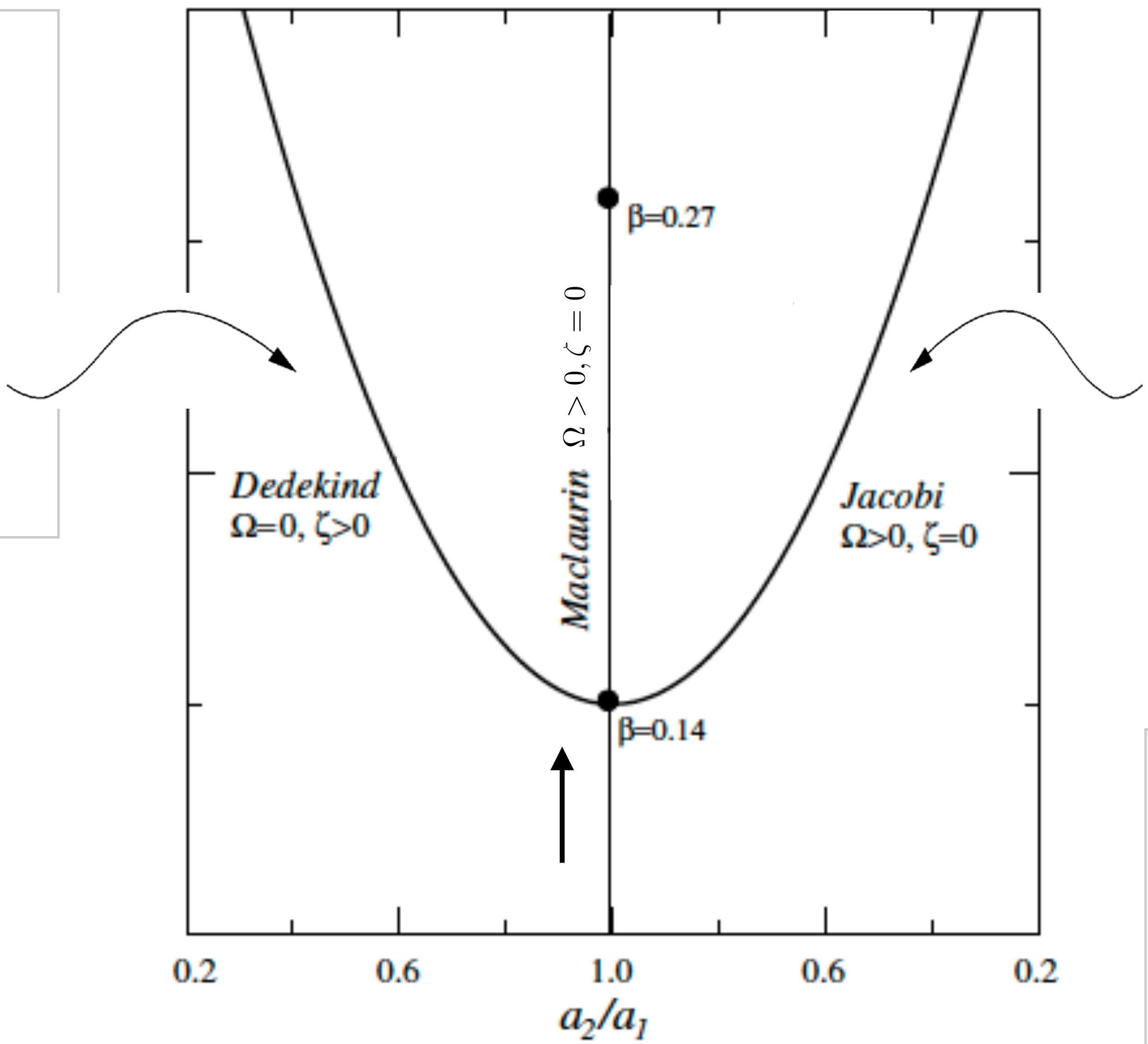
Unstable modes & Ellipsoids (IV)

[Andersson 2003]

GW-driven instability

$J_{\mathrm{Ded}} < J_{\mathrm{Mac}}$

$\mathcal{C}_{\mathrm{Ded}} = \mathcal{C}_{\mathrm{Mac}}$



Viscosity-driven instability

$E_{\mathrm{Jac}} < E_{\mathrm{Mac}}$

$J_{\mathrm{Jac}} = J_{\mathrm{Mac}}$

angular momentum: J

energy: E

circulation: $\mathcal{C} = \oint \mathbf{v} \cdot d\mathbf{l}$

Secular instability: with dissipation, J & D sequences branch out at $\beta_s \approx 0.14$

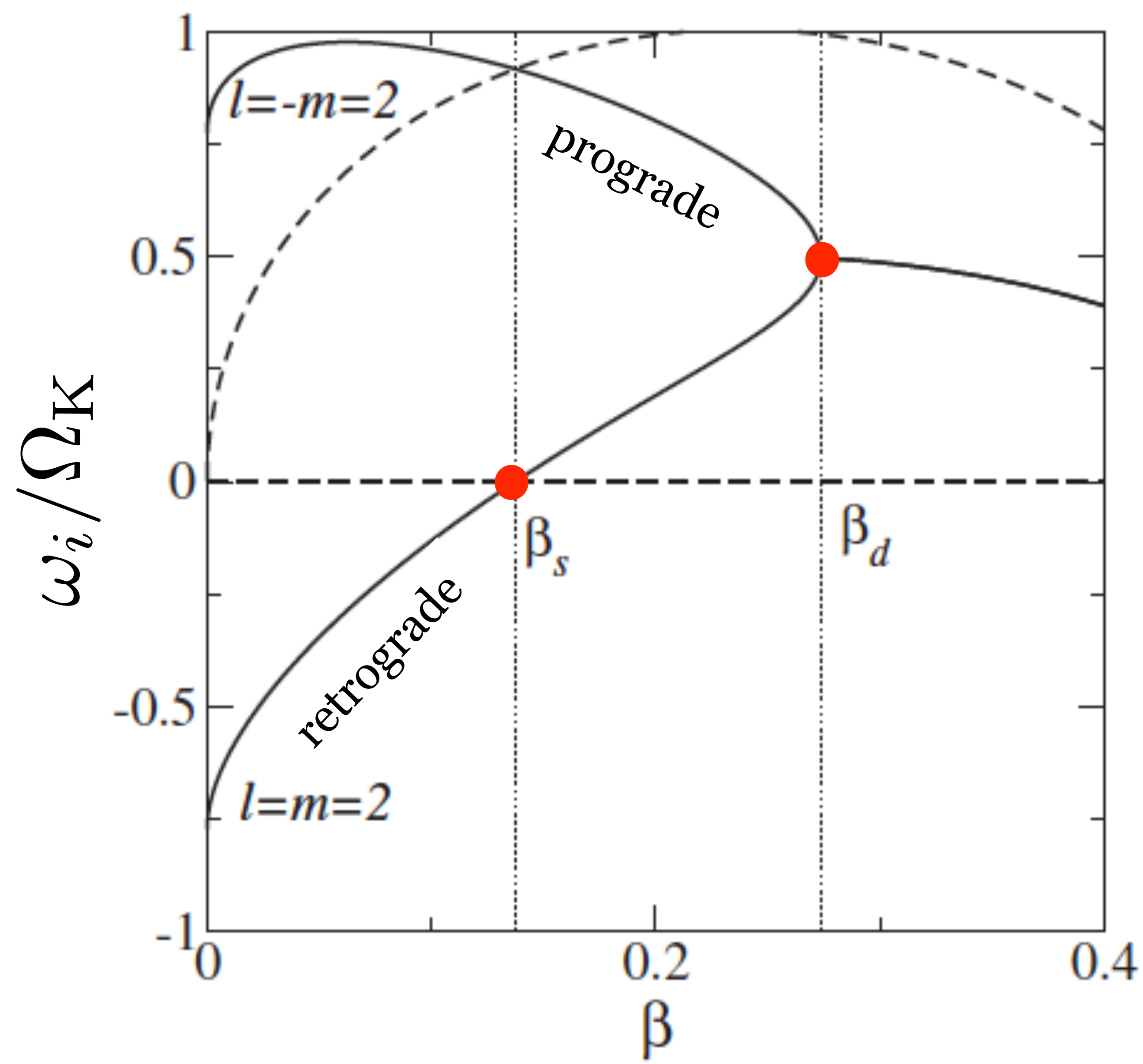
Dynamical instability: the Maclaurin sequence ends at $\beta_d \approx 0.27$

Unstable modes & Ellipsoids (V)

- Ellipsoidal changes are achieved via *unstable* $\ell = |m| = 2$ ("bar") f-modes.

- At β_s :
  retrograde *f*-mode becomes prograde (dragged by stellar rotation)

- At β_d :
  the two *f*-modes merge and become complex-valued.



[Andersson 2003]

mode's *pattern speed*:

$$\omega t + m\varphi = \text{const.}$$

$$\Rightarrow \dot{\varphi} = -\frac{\omega}{m}$$

> 0 for prograde
< 0 for retrograde

Realistic "ellipsoids"

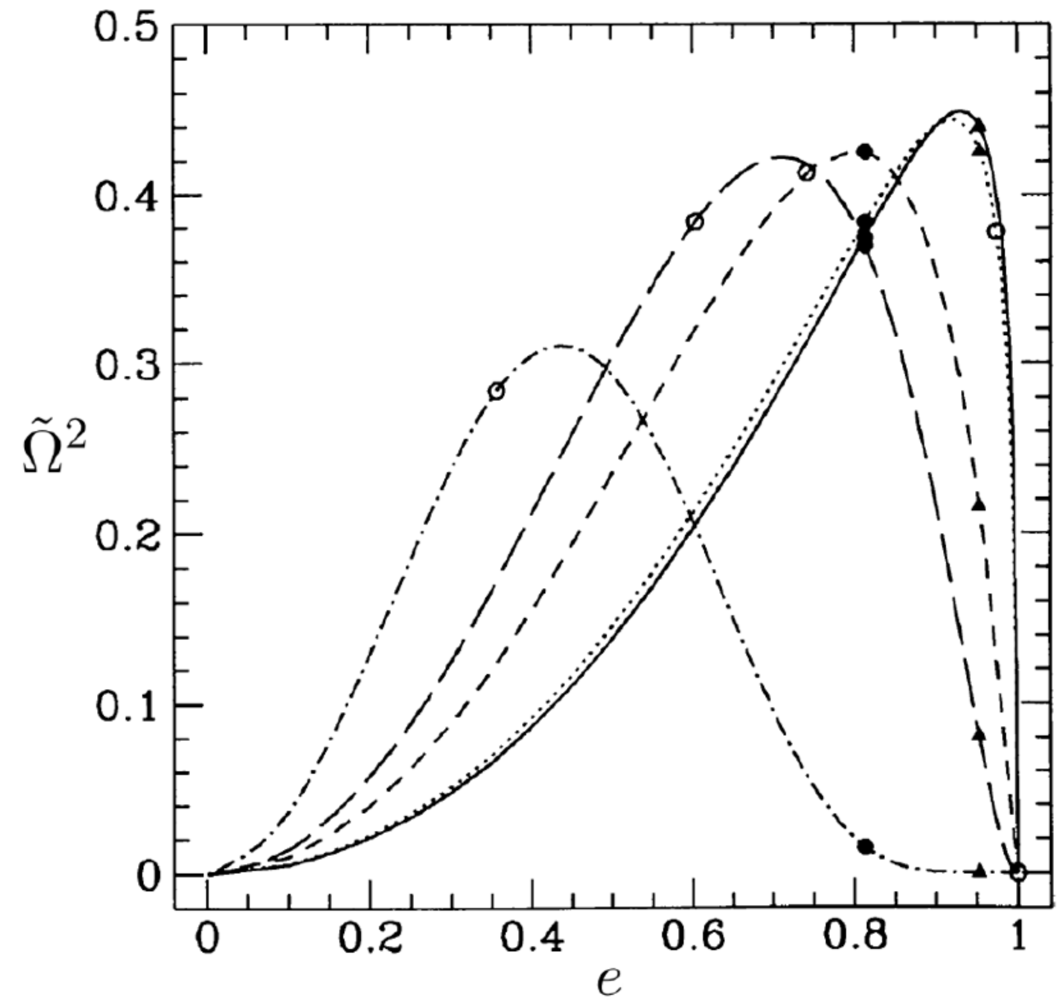
- Realistic (= inhomogeneous, GR gravity) rotating, self-gravitating fluids have a number of important qualitative differences.

- The mass-shedding *Kepler limit*:

  - for uniform bodies, it lies at e=1 (well after the bifurcation point).

  - for realistic systems, it appears before or just after bifurcation.

- Different bifurcation points for the Jacobi and Dedekind sequences.
- GR lowers the values of β_s, β_d
- Need *differential rotation* to reach β_d
- Shape oblate but not perfectly ellipsoidal.
- *Secular instability driven by other modes* (e.g. r-modes).



[Lai 1993]

Unstable *f*-modes in a liquid drop

- No need to look at the stars for observing the *dynamical f*-mode instability!

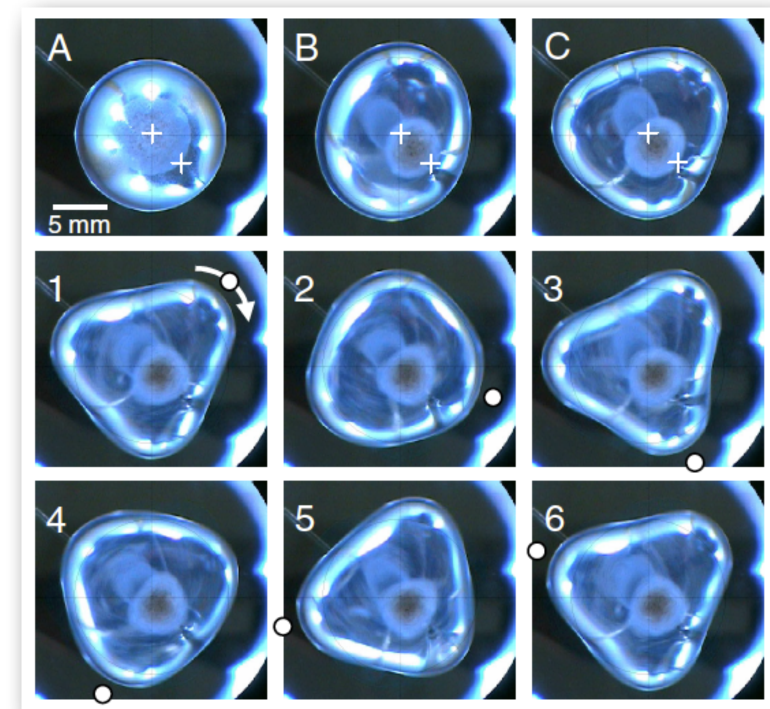
- Rotating liquid drops (suspended by a magnetic field) acquire a series of *n*-lobed "peanut" shapes by the instability of their *f*-modes.

- The drop's *f*-mode is due to its surface tension σ:

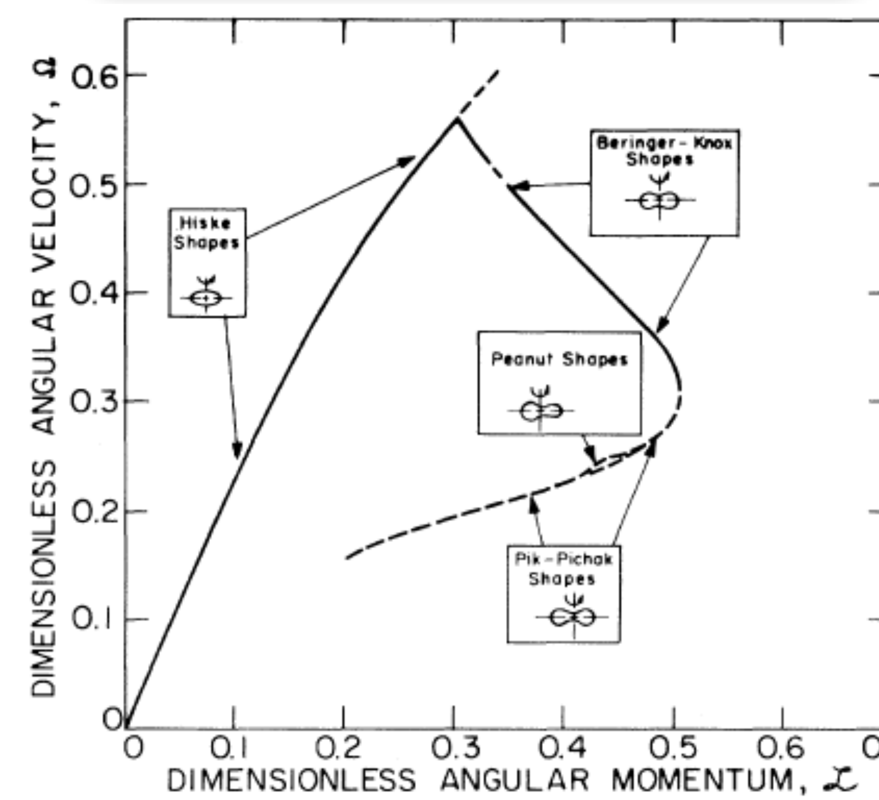
$$\omega \sim \sqrt{\sigma/\rho R^3}$$

- Shape-shifting takes place when rotation exceeds a threshold:

$$\Omega > \{0.56, 0.71, 0.75\} \times \sqrt{\frac{8\sigma}{\rho R^3}}$$



[Hill & Eaves 2008]



[Brown & Scriven 1980]

The CFS instability (I)

- The Chandrasekhar-Friedman-Schutz instability (1970s) is *secular*: the fluid must be coupled to some dissipative mechanism (GWs, EMs, fluid viscosity).

- Quick way to "discover" the GW-CFS instability: formula for GW luminosity

$$\dot{E}_{\rm mode} = -\omega_r \sum_{\ell \geq 2} \omega_i^{2\ell+1} N_\ell \left(\underbrace{|D_{\ell m}|^2}_{\text{mass multipoles}} + \underbrace{|J_{\ell m}|^2}_{\text{current multipoles}} \right)$$

$$\omega_i = \omega_r - m\Omega$$

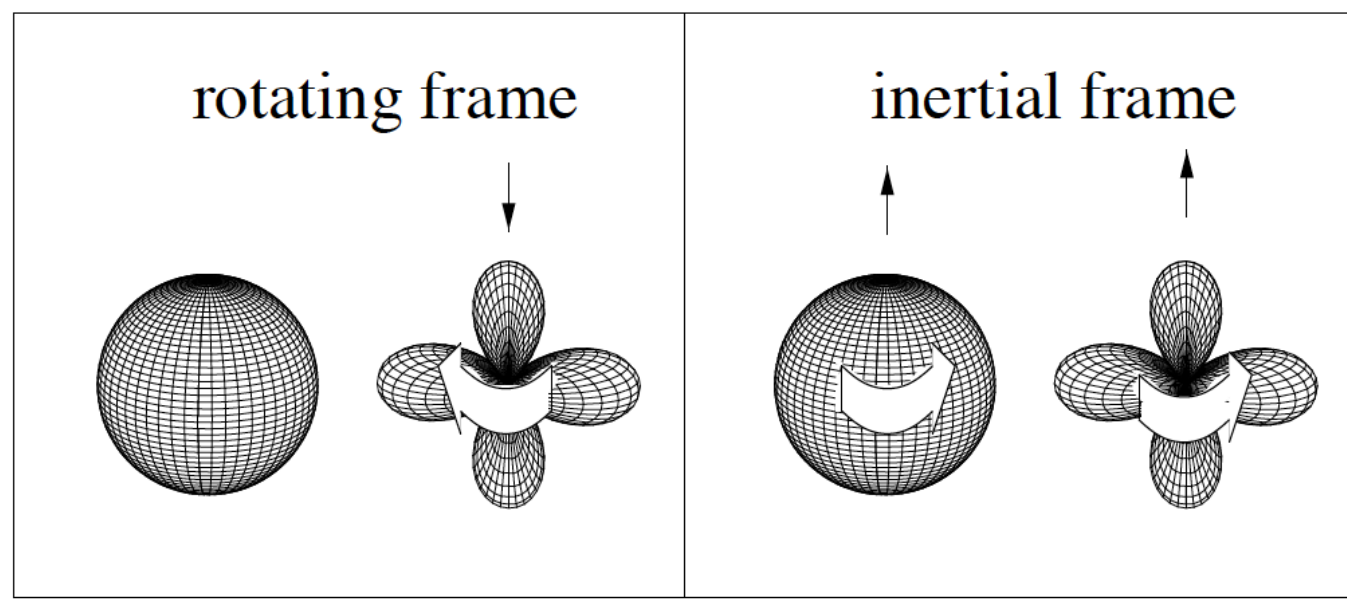
ω_i: inertial frame frequency; ω_r: rotating frame frequency

- *CFS instability*:

$$\dot{E}_{\rm mode} = -\frac{2E_{\rm mode}}{\tau_{\rm gw}} > 0 \;\Leftrightarrow\; \omega_i \omega_r < 0$$

$$\text{For} \quad \omega_r > 0 \;\Rightarrow\; \Omega > \frac{\omega_r}{m}$$

mode "dragged" by stellar rotation



[Andersson & Kokkotas 2001]

CFS theory primer (I)

- The Lagrangian fluid displacement $\boldsymbol{\xi}$ is the main variable:

$$\Delta v^i = \delta v^i + \mathcal{L}_\xi v^i = \dot{\xi}^i$$

- The *inertial frame* Euler equation can be written in the "ABC" form:

$$A^i{}_j\,\ddot{\xi}^j + B^i{}_j\,\dot{\xi}^j + C^i{}_j\,\xi^j = F^i_{\rm diss}$$

- Define the inner product: $\langle \eta^i, \xi_i \rangle \equiv \int dV \bar{\eta}^i \xi_i$

- The *non-dissipative* system admits a conserved *canonical* energy and angular momentum:

$$\xi^i \propto e^{i\omega t + im\varphi}$$

$$E_{\rm c} = \frac{1}{2} m \left[\langle \dot{\xi}^i, A\dot{\xi}_i \rangle + \langle \xi^i, C\xi_i \rangle \right] = \omega \left[\omega \langle \xi^i, A\xi_i \rangle - \frac{i}{2} \langle \xi^i, B\xi_i \rangle \right]$$

$$J_{\rm c} = -\mathrm{Re} \left\langle \partial_\varphi \xi^i, A\dot{\xi}_i + \frac{1}{2} B\xi_i \right\rangle = -m \left[\omega \langle \xi^i, A\xi_i \rangle - \frac{i}{2} \langle \xi^i, B\xi_i \rangle \right]$$

$$\Rightarrow E_{\rm c} = -\frac{\omega}{m} J_{\rm c}$$

CFS theory primer (II)

- *Dynamical* instability: ω complex-valued $\Rightarrow E_{\rm c} = J_{\rm c} = 0$
- *Secular* instability: need $E_{\rm c} < 0$, since $\dot{E}_{\rm c} < 0$ under GW emission.
- Key angular momentum inequality:

$$-\frac{\omega}{m} - \Omega\left(1 + \frac{1}{m}\right) \leq \frac{J_c}{m^2 \langle \xi^i, \rho \xi_i \rangle} \leq -\frac{\omega}{m} - \Omega\left(1 - \frac{1}{m}\right)$$

when the mode's pattern speed changes sign we always have $J_{\rm c} < 0$

$\Longrightarrow$ retrograde mode $\rightarrow$ prograde mode implies $E_{\rm c} < 0$

GW-driven CFS instability condition: $\dfrac{\omega_i}{m} = 0$

- For the viscosity-driven CFS instability a similar *rotating frame* analysis applies and the associated condition is: $\dfrac{\omega_r}{m} = 0$

CFS theory primer (III)

- Growth timescales for the CFS instability:

dynamical: $\tau_{\rm grow} \sim 1/\sqrt{G\rho} \sim \sqrt{R^3/GM}$ ("free-fall" timescale)

secular: controlled by dissipation mechanism (GWs, viscosity) and Ω

- GWs and viscosity are always competing factors. The GW instability is always the dominant one and will be our focus from now on.

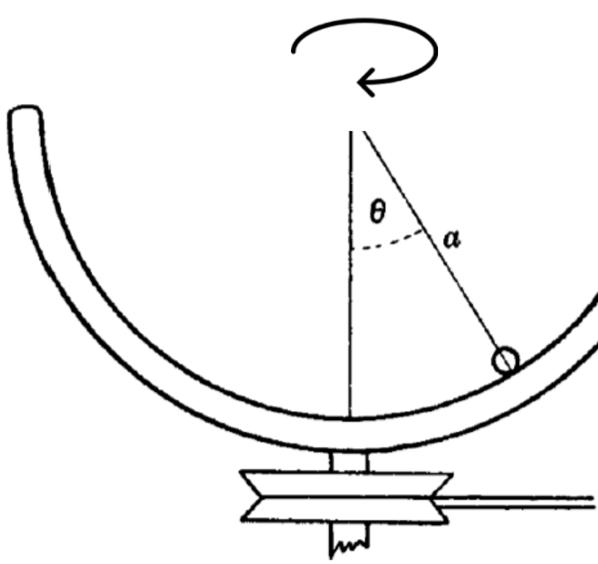
Which modes are easier to CFS-destabilise?

modes (e.g. *f*-mode) with $\omega \neq 0$ @ $\Omega = 0$
can only become unstable above a threshold $\Omega_{\rm cfs}$

trivial modes (e.g. *inertial* modes): $\omega = 0$ @ $\Omega = 0$
with hindsight, we expect these to be the best candidates, perhaps with $\Omega_{\rm cfs} = 0$!

CFS mechanical analogue (I)

- Lamb 1908: particle in rotating bowl, perturbed from equilibrium θ= (x,y) =0.



Lagrangian:

$$\mathcal{L} = \frac{1}{2}ma^2\left[\dot{\theta}^2 + \sin^2\theta\,(\Omega+\dot{\varphi})^2\right] - \underbrace{mga(1-cos\theta)}_{V}$$

equilibrium points:

$$\frac{dV}{d\theta} = 0 \Rightarrow \begin{cases} \theta = 0 \Rightarrow \text{ stable for } \Omega < \sqrt{\frac{g}{a}} \\ \theta = \cos^{-1}\left(\frac{g}{a\Omega^2}\right) \Rightarrow \text{ stable for } \Omega \geq \sqrt{\frac{g}{a}} \end{cases}$$

$\Rightarrow \; \Omega = \sqrt{\frac{g}{a}}$ is a bifurcation point

CFS mechanical analogue (II)

- After adding friction, the motion near θ= (x,y) =0 is described by (rotating frame):

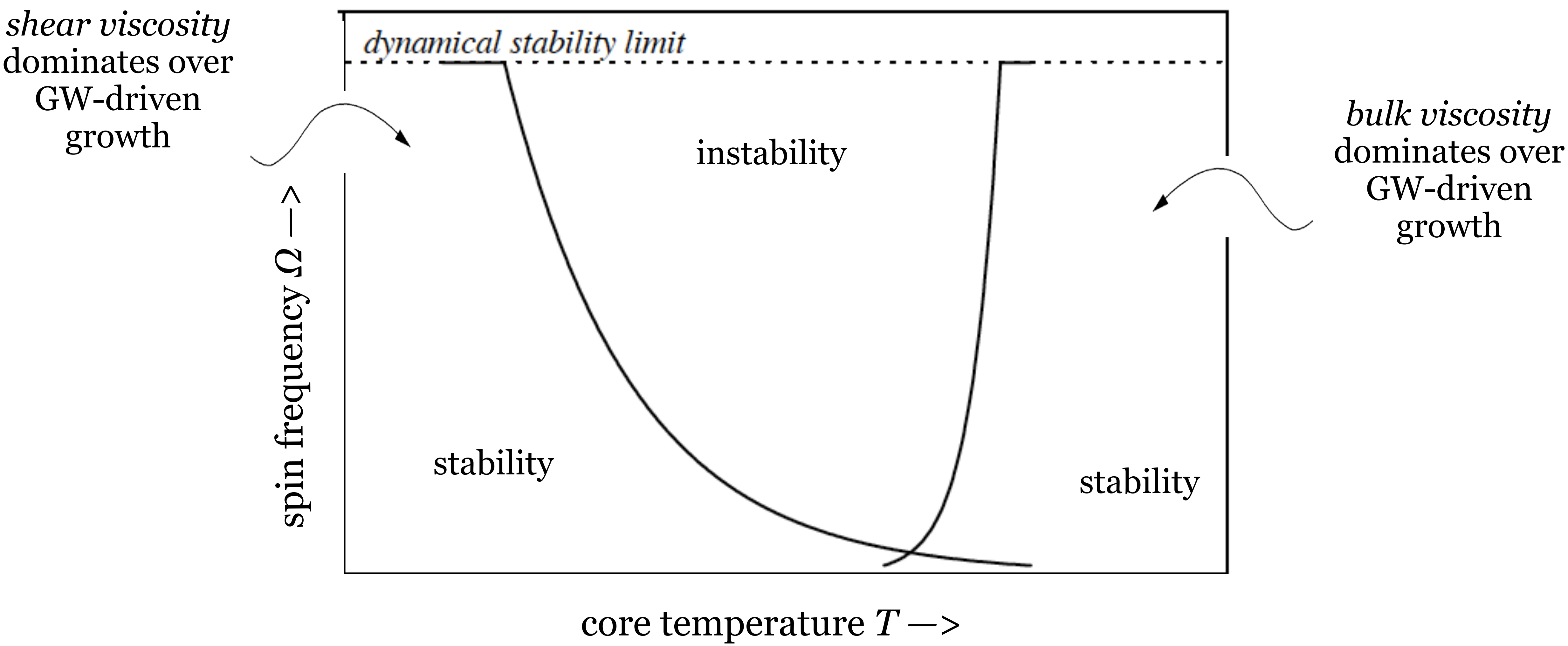
$$\zeta(t) = x(t) + iy(t) \qquad \ddot{\zeta} + (\,2i\Omega + \underset{\text{friction}}{\lambda}\,)\dot{\zeta} + \left(\frac{g}{a} - \Omega^2\right)\zeta = 0$$

general solution: $\zeta(t) = A\, e^{-i(\Omega \mp \sqrt{g/a})t} e^{-\frac{1}{2}\lambda(1 \mp \Omega\sqrt{a/g})t}$

- Motion *unstable* for: $\lambda > 0$ and $\Omega > \sqrt{g/a}$

- This is an example of a *viscosity-driven* "CFS" instability.

The (typical) CFS instability window



instability curve:

$$\frac{1}{\tau_{\rm grow}} + \frac{1}{\tau_{\rm diss}} = 0 \quad \text{where} \quad \frac{1}{\tau_{\rm diss}} = \frac{1}{\tau_{\rm sv}} + \frac{1}{\tau_{\rm bv}} + \ldots$$

f-modes: secular instability

- The *f*-mode is a powerful emitter of GWs (via the mass multipoles) but only becomes unstable at fast rotation: $\Omega > \Omega_{\rm cfs} \sim 0.9\,\Omega_{\rm K}$

- The instability is active in the high-*T* regime, appropriate for newborn NSs. At lower *T* (appropriate for mature NSs), it is suppressed by *superfluid vortex mutual friction*.

- The growth timescale is a steep function of the spin difference $\Omega - \Omega_{\rm cfs}$ and the stellar compactness M/R.

Approximate Newtonian results:

$$\begin{cases} \tau_{\rm gw}(\Omega=0) = f_\ell \left(\dfrac{c^2 R}{GM}\right)^{\ell+1} \dfrac{R}{c} & \ell = m \\ \tau_{\rm gw}(\Omega=0) \approx 0.07\, M_{1.4}^{-3}\, R_6^4\, \mathrm{s} & \ell = m = 2 \\ \tau_{\rm gw}(\Omega) \approx \tau_{\rm gw}(0)\left(1 - \sqrt{\dfrac{m}{3}\dfrac{\Omega}{\Omega_{\rm K}}}\right)^{-2m-1} & \end{cases}$$

- GR is expected to make a big difference in the CFS timescale.

f-modes: instability window

- Recent GR calculations:

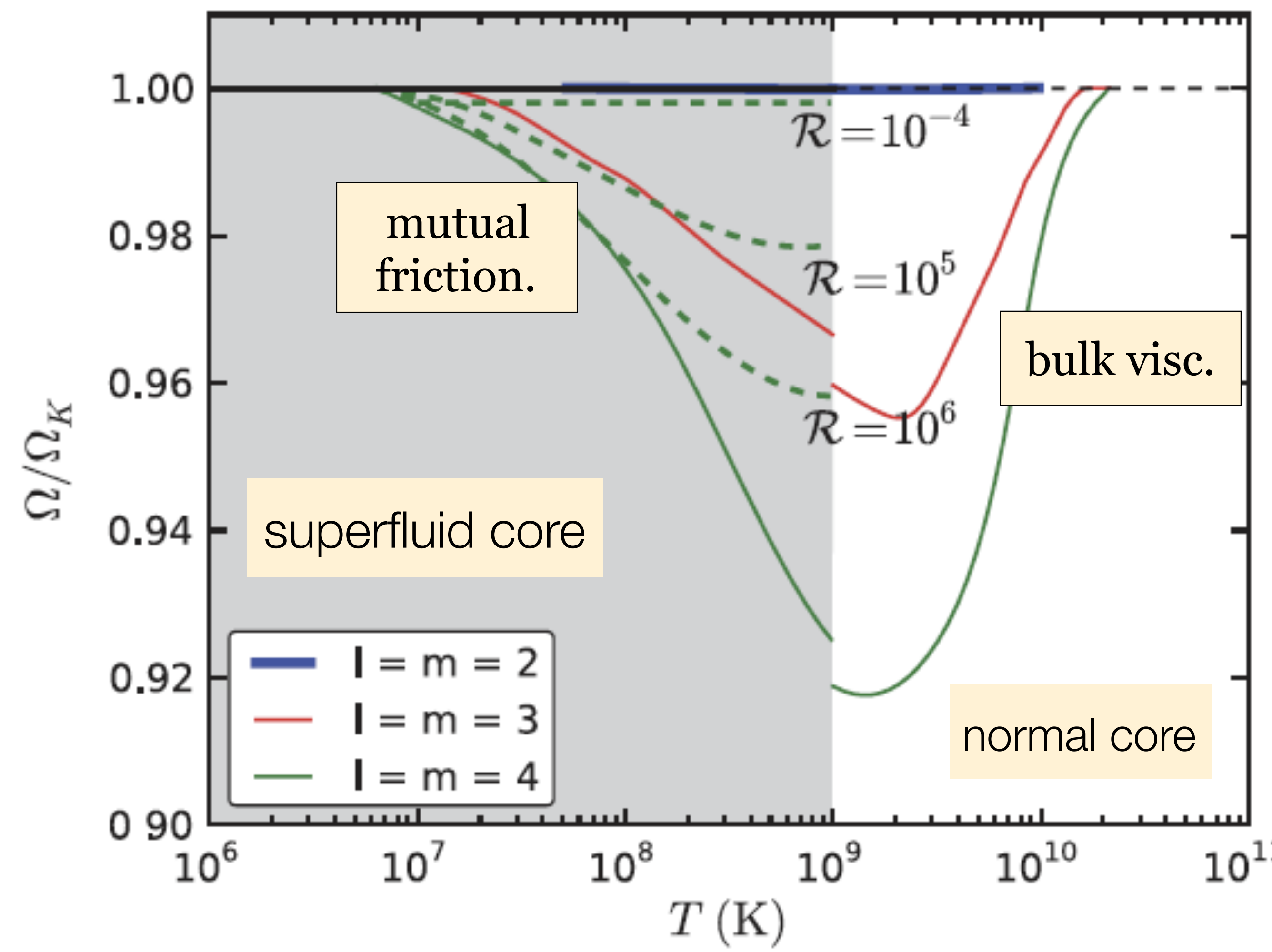
$$\tau_{\rm grow} \sim 10^4 - 10^6 \text{ s}$$

(this is a factor ~ 10 shorter than earlier Newtonian results).

- Typically, the $\ell = m = 4$ mode is the most unstable one.

- Instability *enhanced* in *massive* NSs (shorter growth timescale, larger instability window).

- Damping: bulk viscosity (high T), superfluid vortex mutual friction (low T).



[Gaertig et al. 2011]

Stellar model:

N = 0.73 polytrope

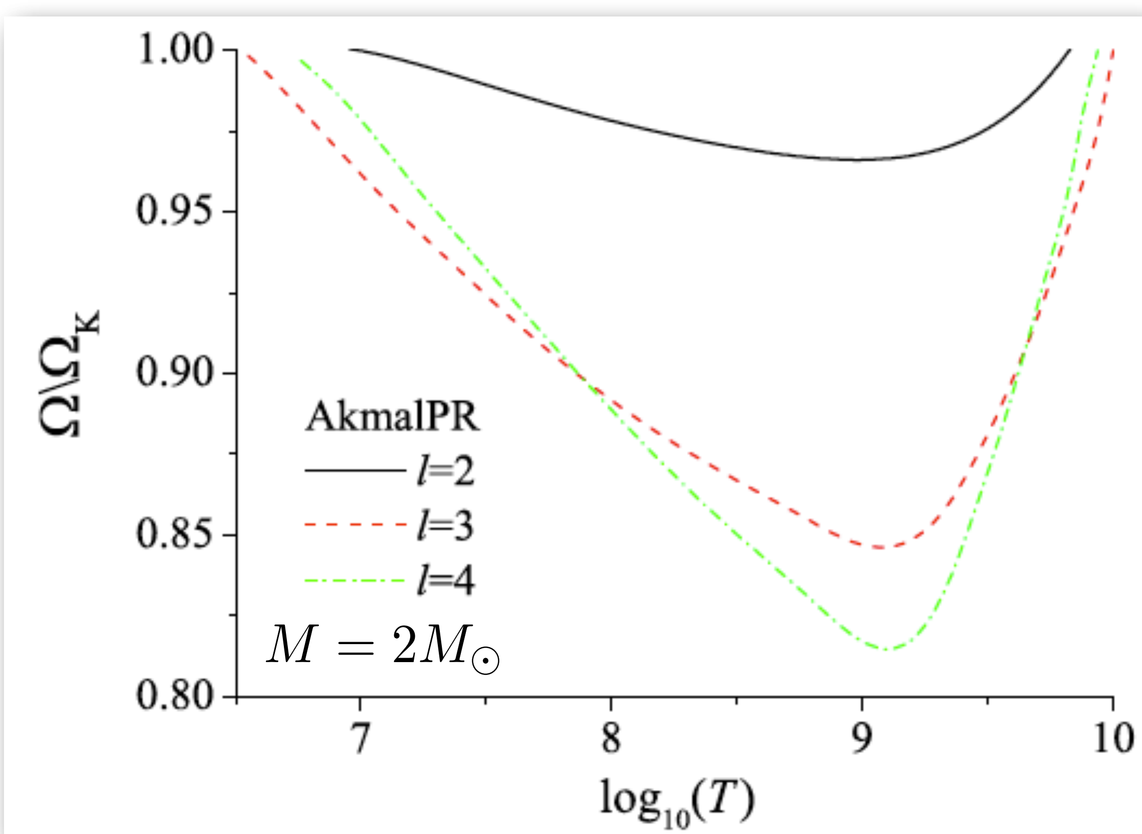
$M = 1.48 M_\odot, \quad R = 10.47\,\text{km}$

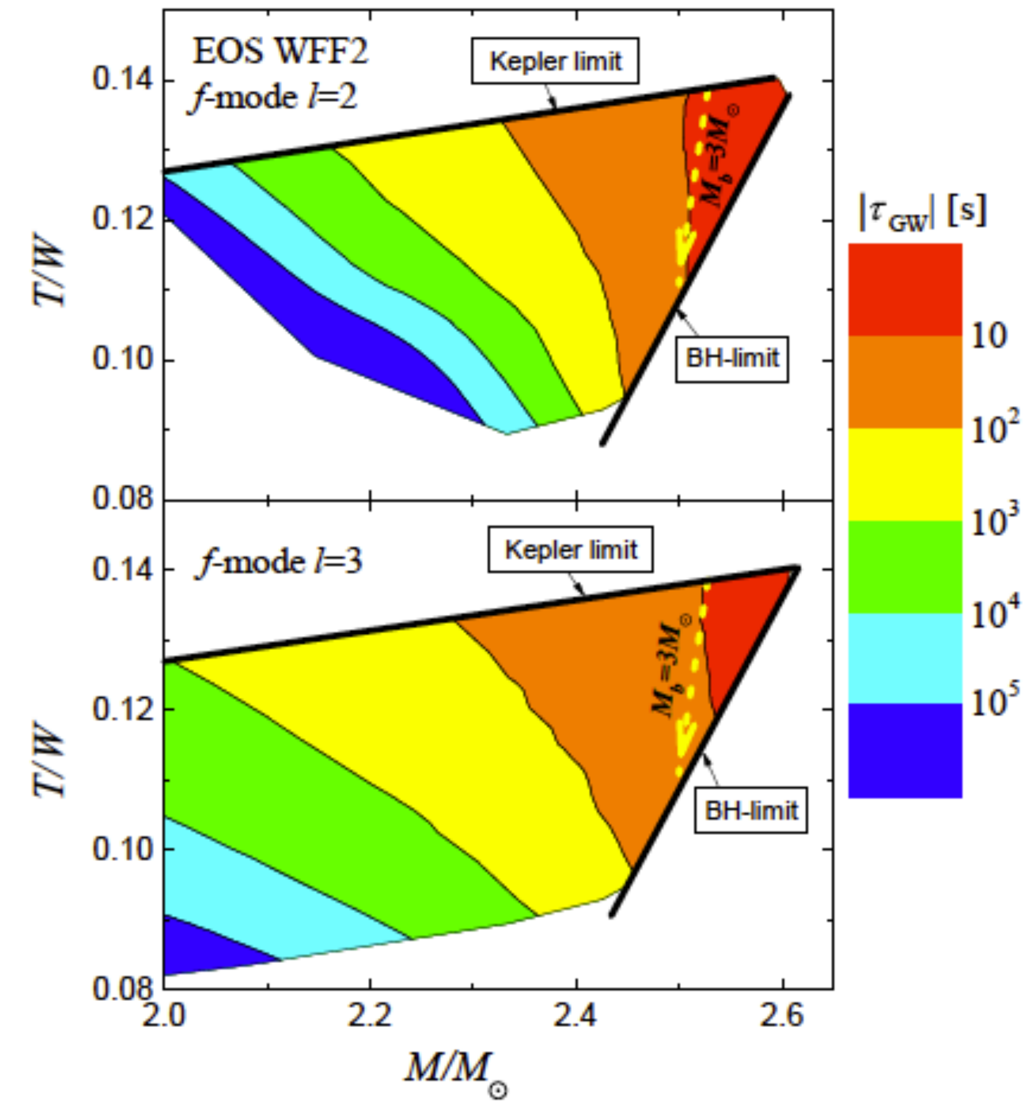
f-modes: (supra)-massive NS

- An optimal arena for the f-mode instability could be a massive NS ($M \gtrsim 2M_\odot$) formed in a NS-NS merger.

- Revised growth timescale:

$$\tau_{\rm grow} \sim 10 - 100\,{\rm s}$$



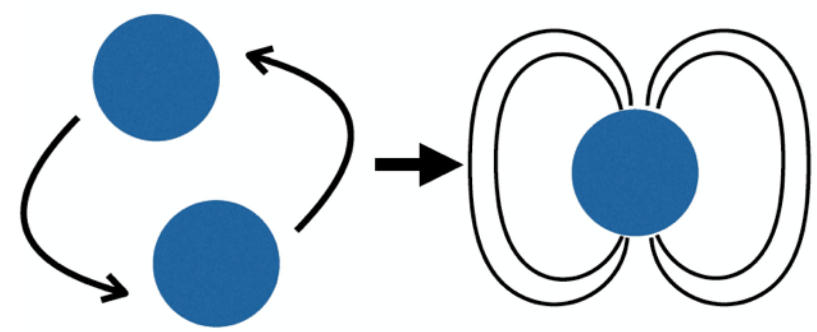


[Doneva et al. 2013, 2015]

f-modes: GW afterglow in sGRBs

- NS-NS mergers likely to produce sGRBs and f-mode-unstable massive NSs

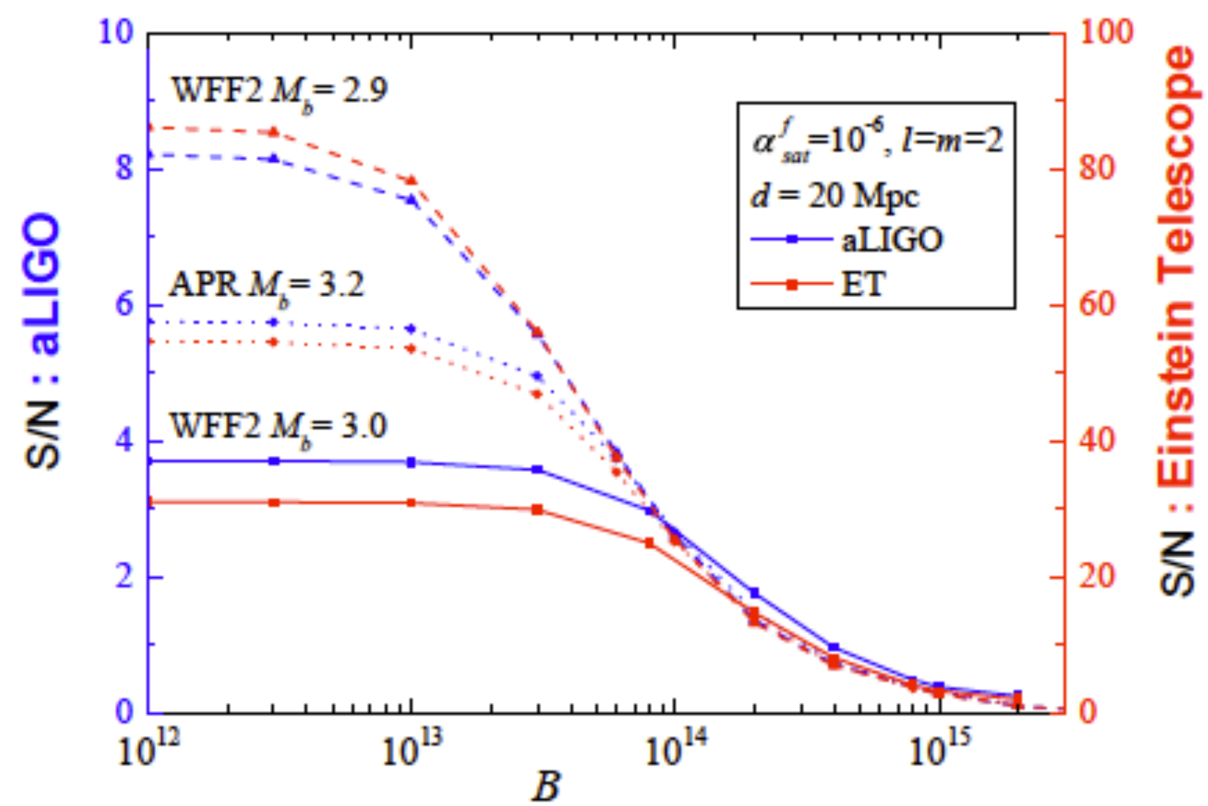
- The f-mode competes against magnetic dipole spin down (and, possibly, unstable r-modes).



- Recent Newtonian result for mode's saturation energy:

$$E_{\mathrm{mode}} \sim (10^{-6} - 10^{-5}) M c^2$$

- f-mode signal could be *detectable* by ET (or by LIGO, if we invoke distances much shorter than those associated with observed sGRBs).



[Doneva et al. 2015]

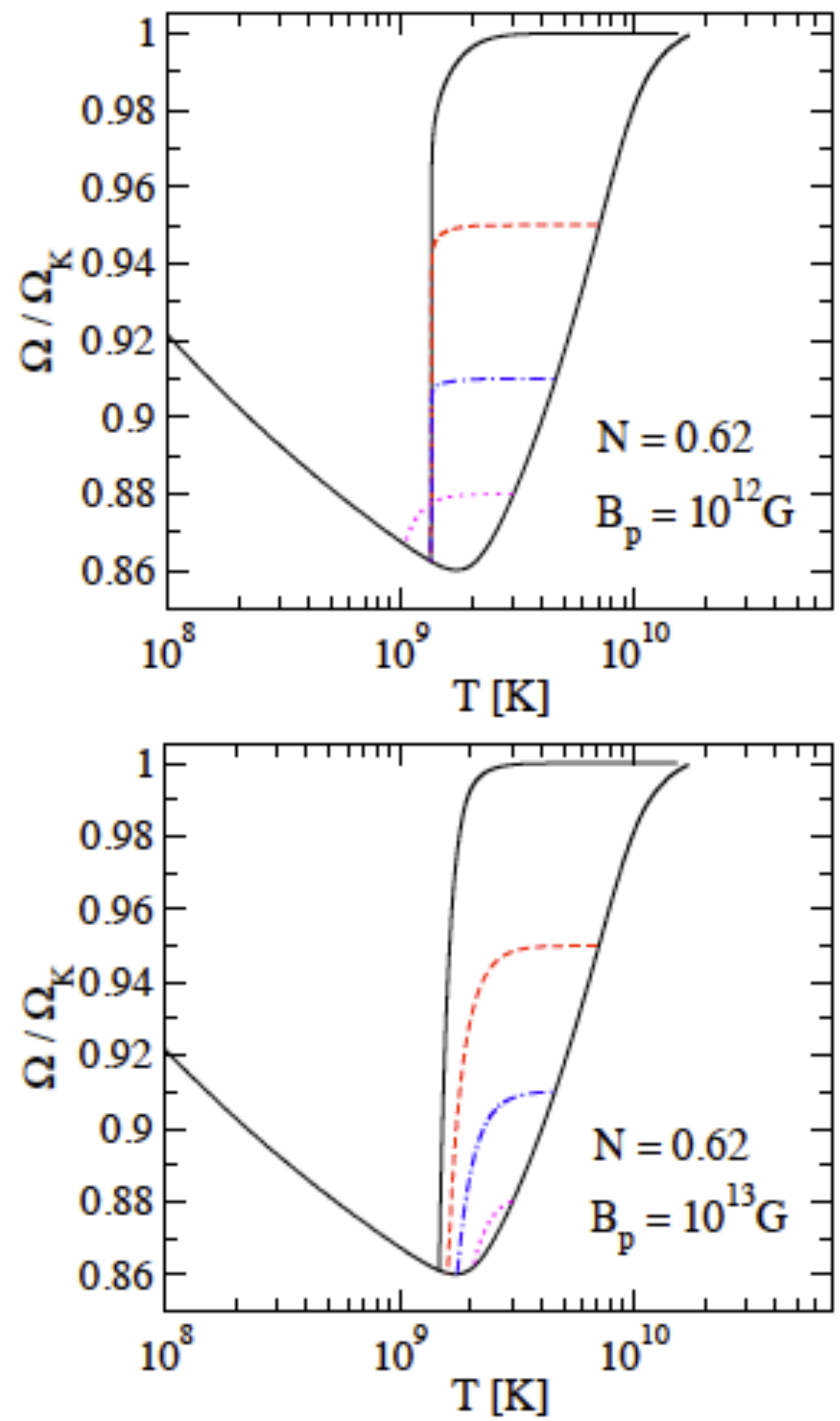
f-modes: spin-temperature evolution

- A rapidly spinning newly-formed NS, undergoes a coupled Ω-T evolution, under the combined action of the f-mode instability & magnetic dipole spin down.

- The GW-driven spin down begins once the mode saturates.

- Calculations suggest a mode growth & spin down before (say) the onset of superfluidity.

- Figure: Newtonian star, $M = 2M_{\odot}$ mode $\ell = m = 4$

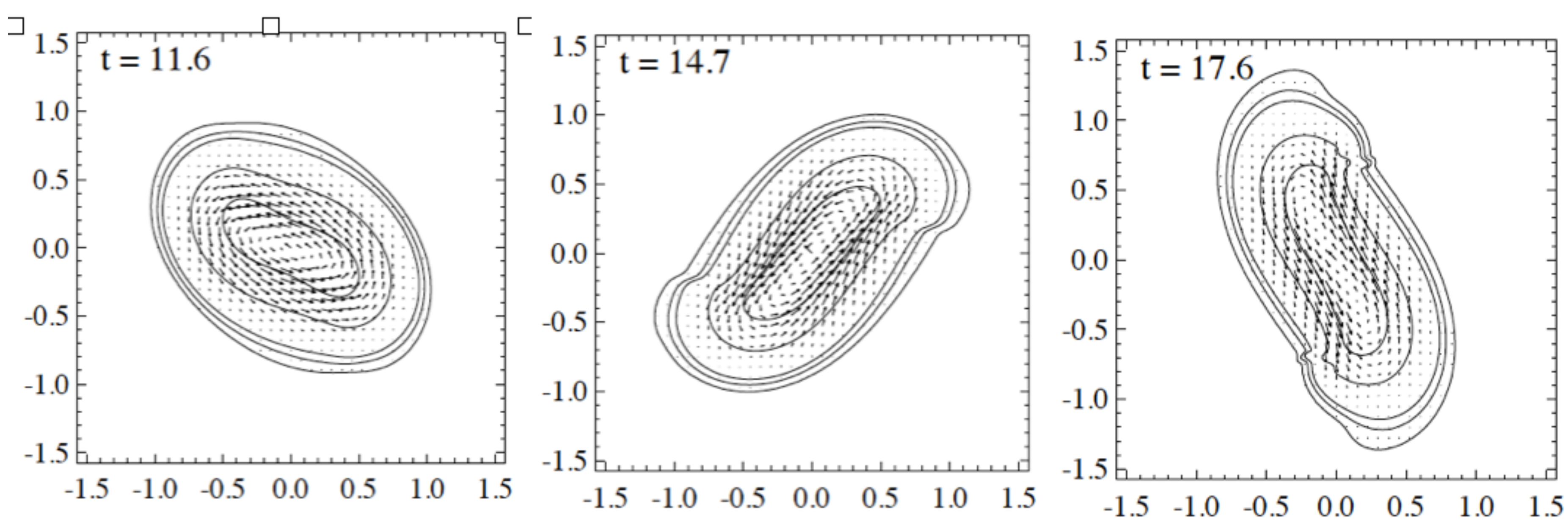


[Passamonti et al. 2013]

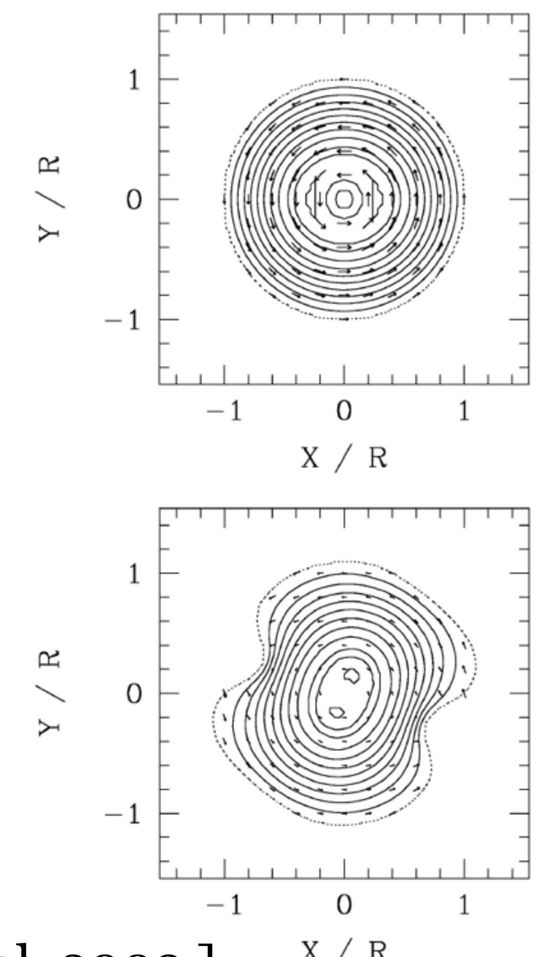
The dynamical f-mode instability

- The dynamical f-mode instability is typically seen in action in the aftermath of NS-NS mergers, where a differentially rotating (supra-)massive NS may form with initial $\beta > \beta_d$

- The f-mode "bar" is formed and copiously radiates GWs until the star is spun down/differential rotation is quenched.



[Cazes & Tohline 2002]



[Shibata et al. 2002]

The r-mode instability

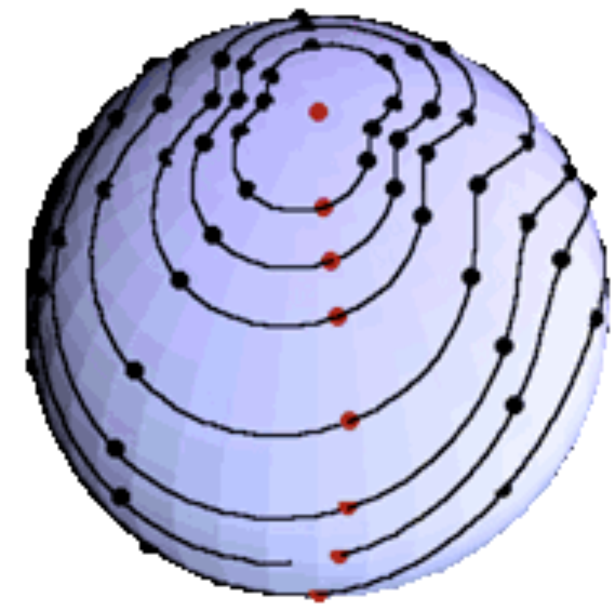
- The r-modes are purely axial inertial modes, characterised by nearly horizontal fluid motion.

- The r-modes are CFS-unstable for *any* spin Ω, i.e. they always have $E_c < 0$.

- They are "special" GW sources as they principally radiate via the current multipoles.

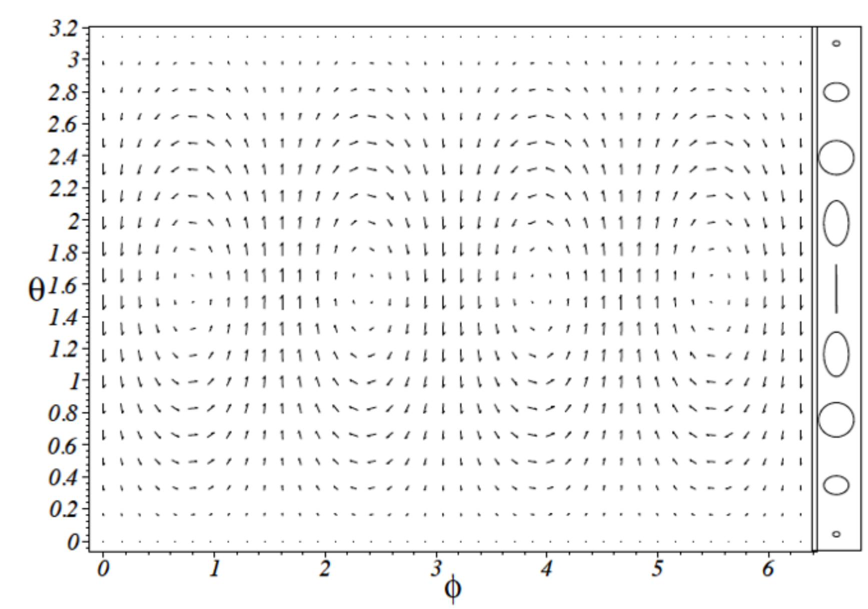
- The $\ell = m = 2$ r-mode is the most unstable one, with growth timescale:

$$\tau_{\rm grow} \approx 40\, M_{1.4}^{-1} R_6^{-4} \left(\frac{P}{1\text{ ms}}\right)^6 \text{ s}$$



r-mode flow
(corotating frame)

[Figure: Hanna & Owen]



[Andersson & Kokkotas 2001]

r-modes: back of the envelope

- The "minimal" stellar model supporting r-modes:

uniform density+Cowling	incompressible flow	slow rotation approximation
$\delta\rho = \delta\Phi = 0$	$\boldsymbol{\nabla}\cdot\delta\mathbf{v} = 0$	$\Omega/\Omega_{\mathrm{K}} \ll 1, \;\; \mathcal{O}(\Omega^2)$ terms $= 0$

Axial parity mode:

$$\delta\mathbf{v} = U_\ell(r)[\,\mathbf{r}\times\boldsymbol{\nabla}Y_\ell^m\,]e^{i\omega t}$$

Euler (rotating frame):

$$\mathbf{E} = \partial_t\delta\mathbf{v} + 2\boldsymbol{\Omega}\times\delta\mathbf{v} + \frac{1}{\rho}\boldsymbol{\nabla}\delta p = 0$$

Euler "combos":

$$\left\{\begin{array}{l} \boldsymbol{\nabla}\cdot\mathbf{E} = 0 \\ \\ \hat{\mathbf{r}}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \sin\theta\,\partial_\theta E^\theta - mE^\varphi = 0 \\ \\ mE^\theta - \sin\theta\,\partial_\theta E^\varphi = 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \omega\,[\,\ell(\ell+1)\omega - 2m\Omega\,] = 0 \;\Rightarrow\; \underline{\omega = \dfrac{2m\Omega}{\ell(\ell+1)} = \dfrac{2\Omega}{m+1}} \\ \\ rU_m' - (m+1)U_m = 0 \;\Rightarrow\; \underline{U_m = Ar^{m+1}} \qquad \underline{\ell = m} \end{array}\right.$$

r-modes: "exact" calculation

Fast rotation effects
($\mathcal{O}(\Omega^2)$ and higher)

no purely axial solutions, instead
we have "*axial-led*" inertial r-modes

$$\underbrace{|U_\ell|}_{\mathcal{O}(\Omega)} \gg \underbrace{|W_\ell|, |V_\ell|}_{\mathcal{O}(\Omega^2)}$$

$$\delta\rho,\ \nabla \cdot \delta\mathbf{v} \sim \mathcal{O}(\Omega^2)$$

coupling to higher multipoles

$$U_{\ell=m} \begin{cases} \{U_{m+2}, U_{m+4}, \ldots\} \\ \{W_{m+1}, V_{m+1}, W_{m+3}, V_{m+3}, \ldots\} \end{cases}$$

GR effects
(slow rotation $\mathcal{O}(\Omega)$)

axial-led r-modes & multipole
coupling (as above)

$$\underbrace{|U_\ell|}_{\mathcal{O}(1)} \gg \underbrace{|W_\ell|, |V_\ell|}_{\mathcal{O}(M/R)}$$

minor modification in GW growth timescale

post-Newtonian frequency
(uniform density model)

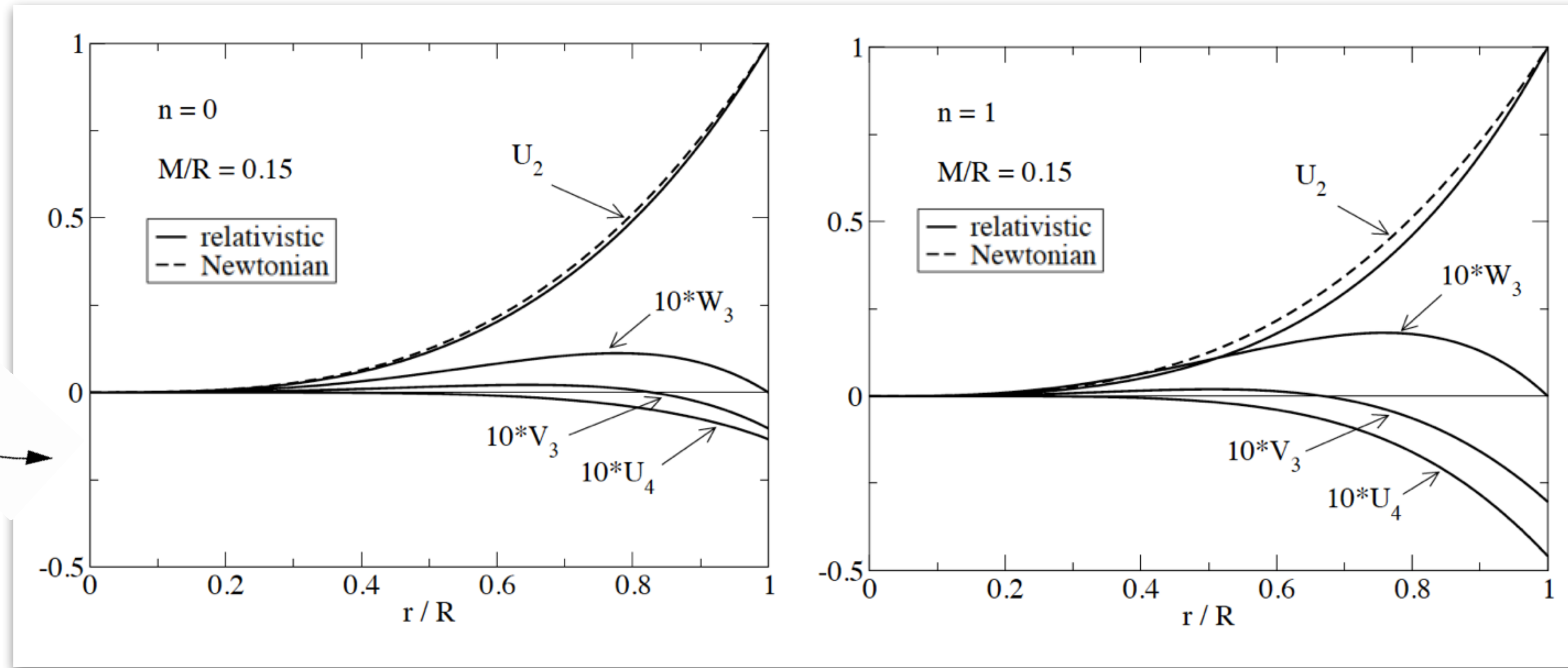
$$\omega = \frac{2m\Omega}{\ell(\ell+1)}\left[1 - \kappa_m \frac{M}{R} + \mathcal{O}\left(\frac{M}{R}\right)^2\right]$$

redshift/frame
dragging correction:

$$\kappa_m = \frac{8}{5}\frac{(m-1)(2m+11)}{(2m+1)(2m+5)}$$

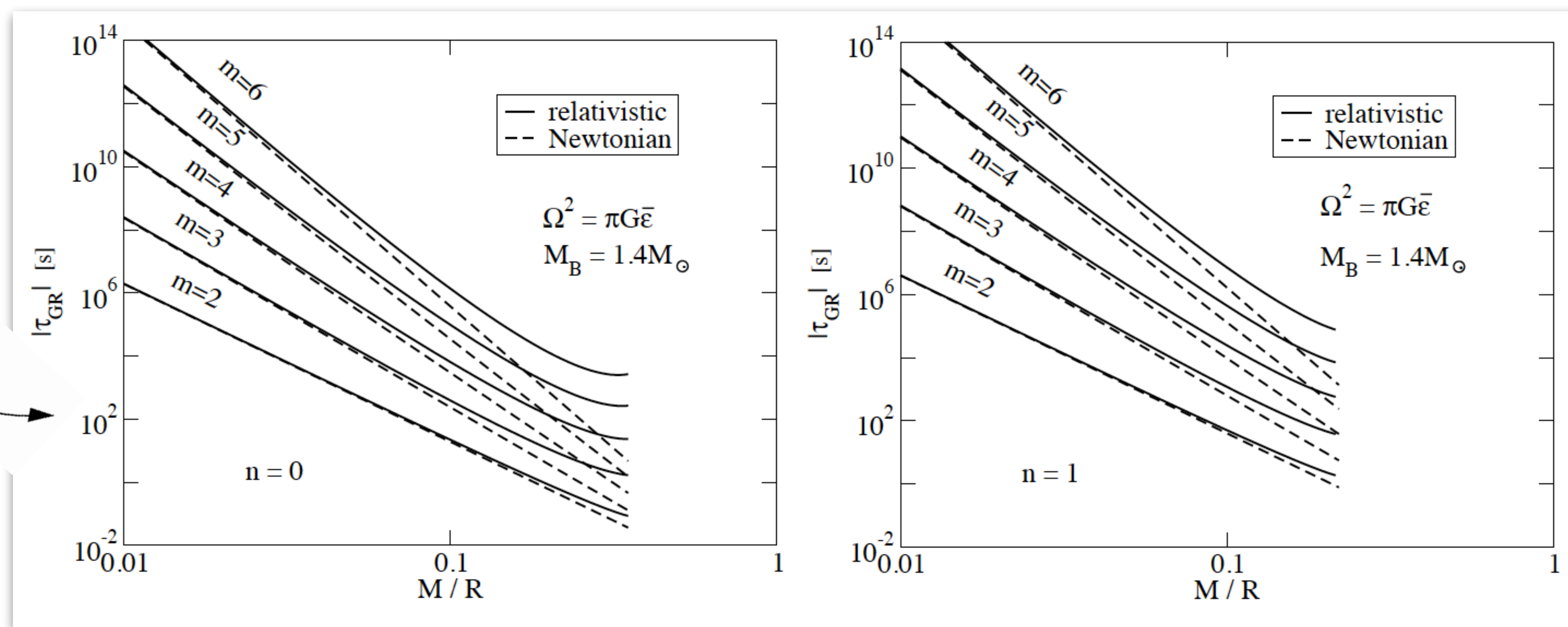
r-modes: sample GR results

Eigenfunctions for m=2 r-mode



n = {0,1} polytropic stars

GW growth timescales

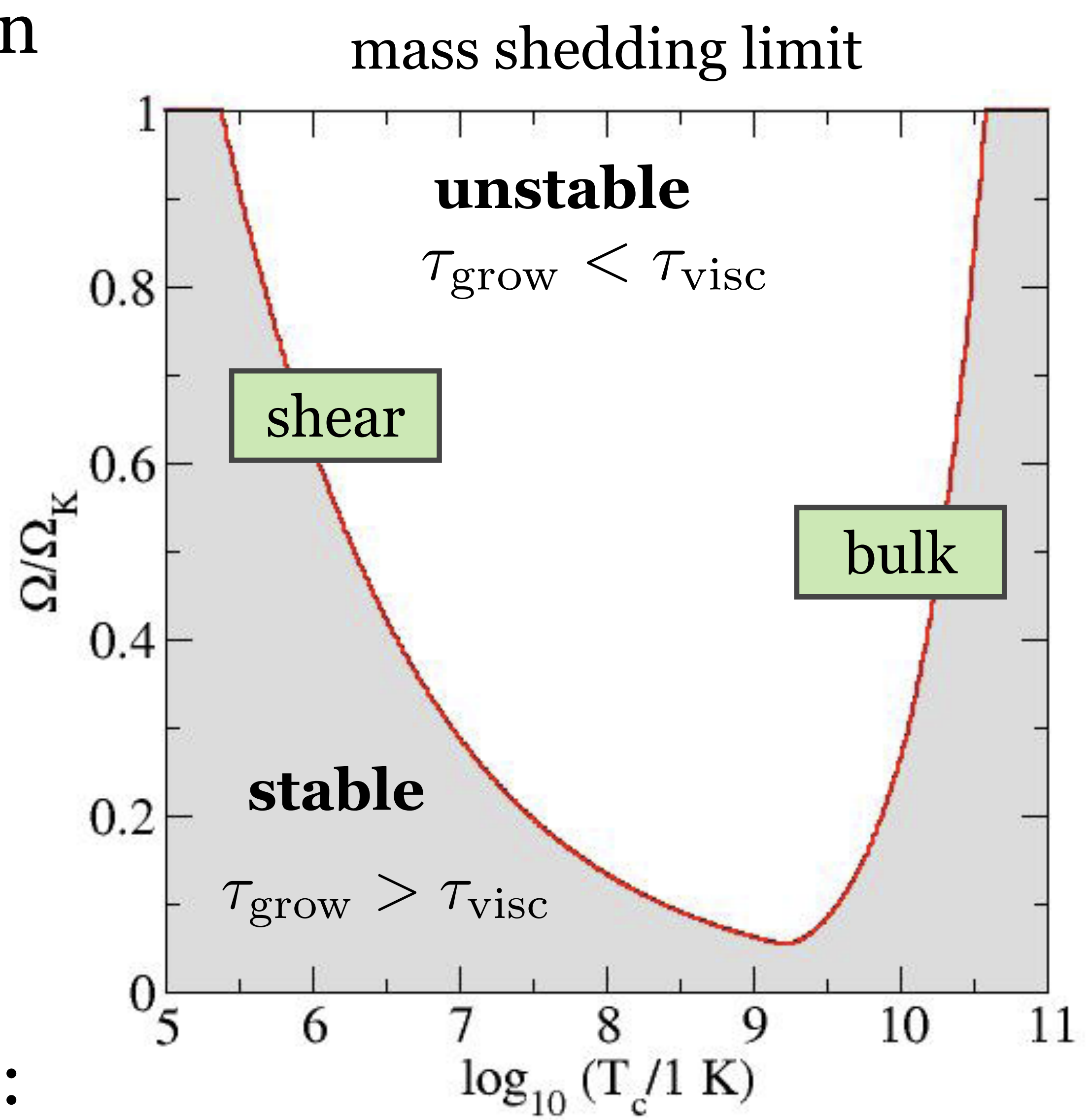


[Lockitch et al. 2003]

r-modes: instability window

- The r-mode instability is active for any rotation but can be damped by viscous processes.
- The spin-temperature instability window is "large" but depends on *uncertain* core-physics.
- "Minimal" model: accounts for damping due to shear and bulk viscosity.
- Once the instability is active, the GW signal is largely determined by the mode's *amplitude* α:

$$\delta v \sim \alpha \left(\frac{r}{R}\right)^2 \Omega R$$



r-modes: how large can they grow?

- Several mechanisms could limit the r-mode's growth, thus saturating its amplitude:

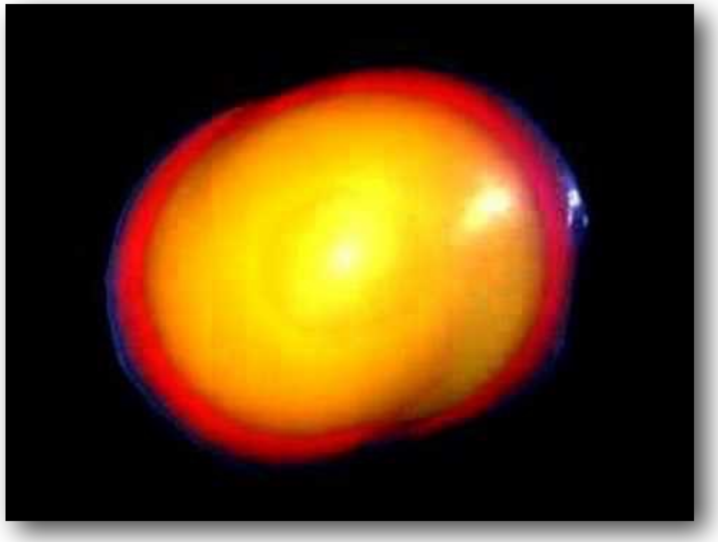
  - Non-linear coupling with short-wavelength modes (mostly inertial):

$$\alpha_{\rm sat} \sim 10^{-4} - 10^{-3}$$

  - Dissipative "cutting" of proton flux tubes by neutron vortices:

$$\alpha_{\rm sat} \sim 10^{-6} - 10^{-5}$$

  - Still under investigation:
    winding up of magnetic field lines by r-mode flow.



r-modes: spin-down upper limits

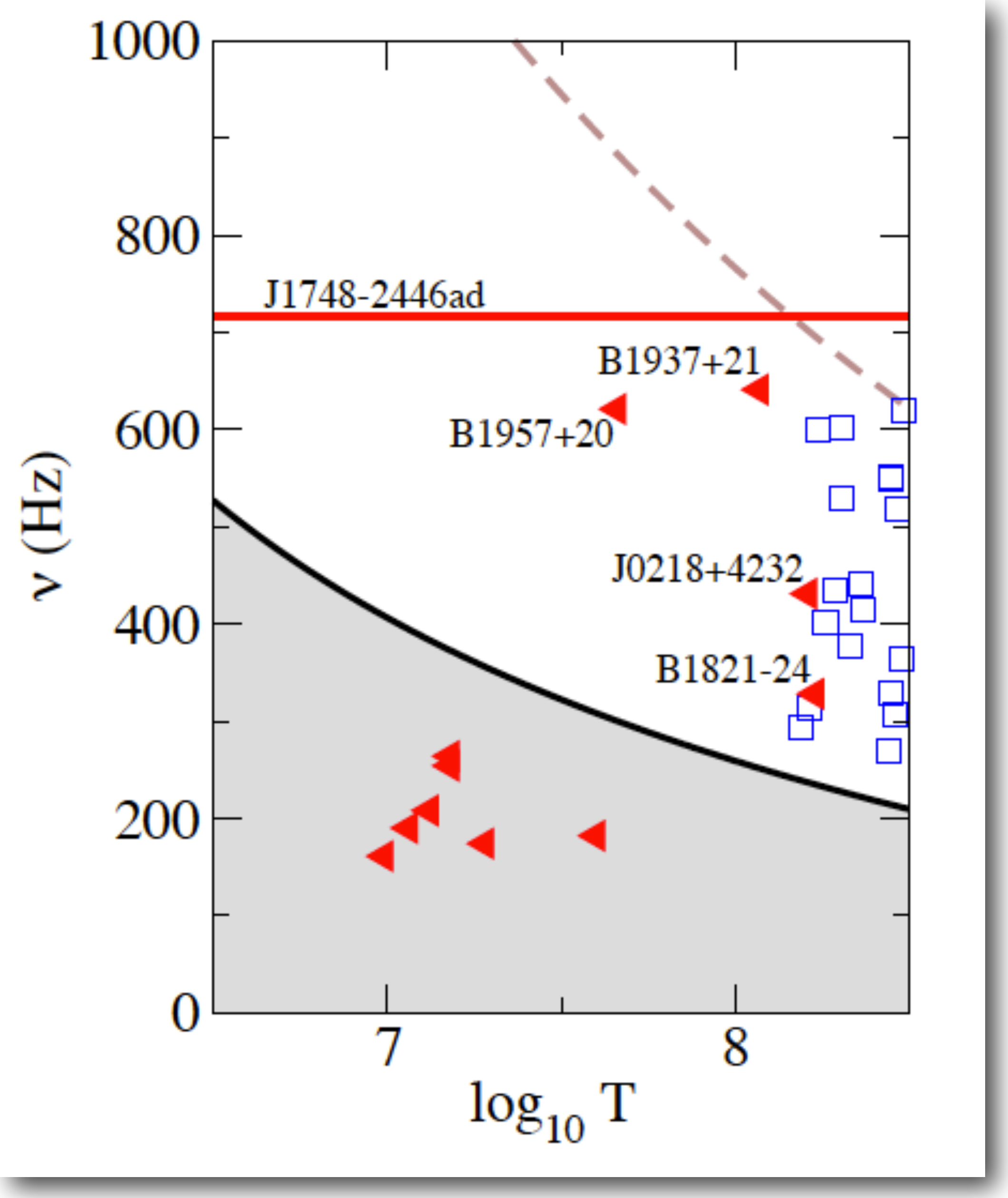
- Assume a “minimum-physics” instability window.

- Then, several LMXBs and MSPs with measured $f_{\rm spin}, \dot{f}_{\rm spin}$ are potentially r-mode unstable.

- Obtain upper limits for the amplitude by assuming spin down only via r-mode GW radiation. The outcome is tiny:

$$\alpha_{\rm sat} \lesssim 10^{-7}$$

- This of course assumes that the systems are r-mode unstable in the first place.



blue: LMXBs
red: MSPs (T data: upper limits)

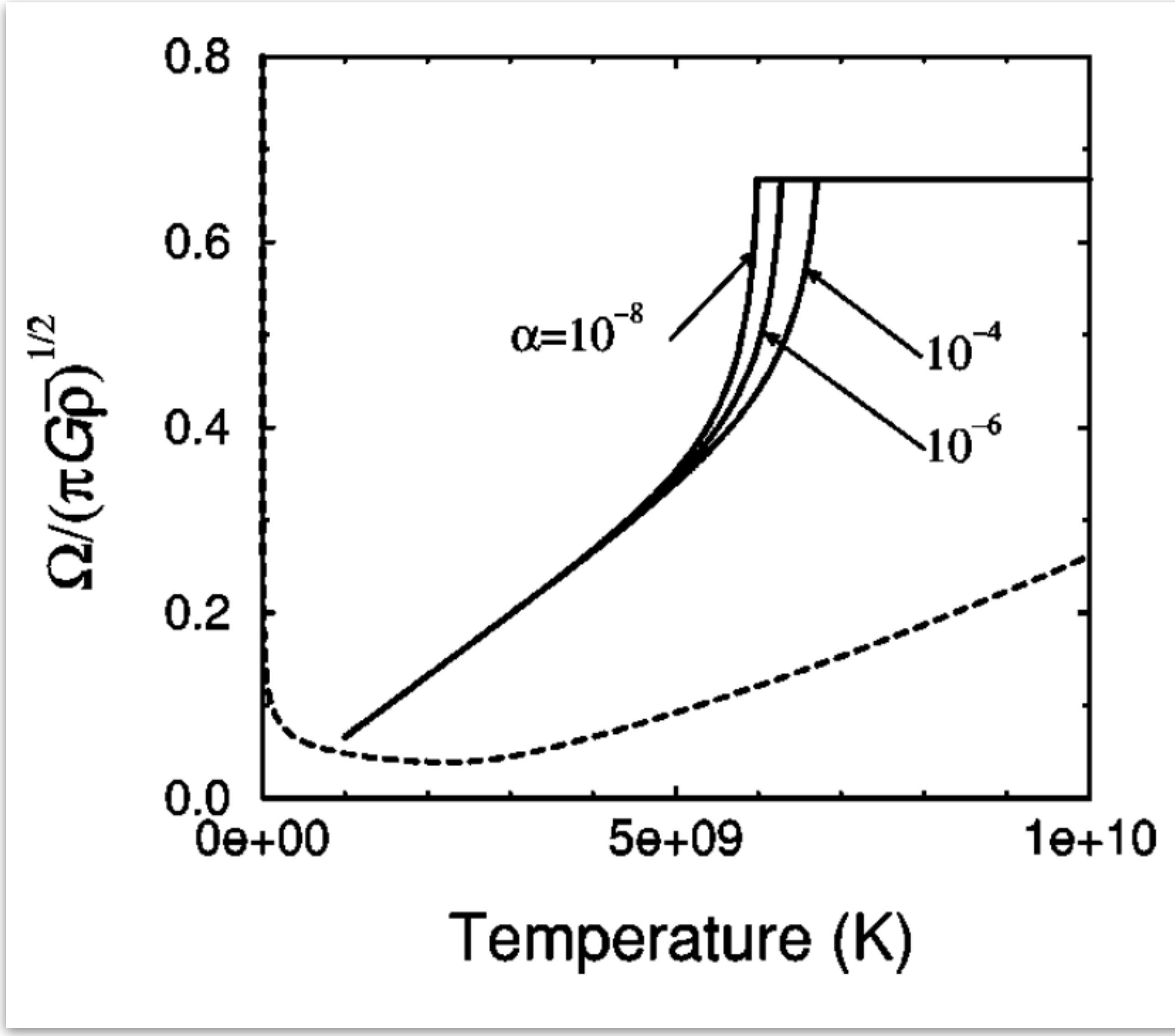
[Figure: N. Andersson]

r-mode evolution of young NSs

- The *r*-mode spin-temperature evolution consists of two phases:

- “linear” phase : mode grows under CFS instability, radiating GWs. Meanwhile, the star cools down but does not spin down significantly.

- “non-linear” phase: the mode saturates ($\alpha = \alpha_{\rm sat}$) and GW emission comes at the expense of the stellar rotational kinetic energy.



[Owen et al. 1998]

r-mode astrophysics: LMXBs

- Spin distribution of NSs in LMXBs:

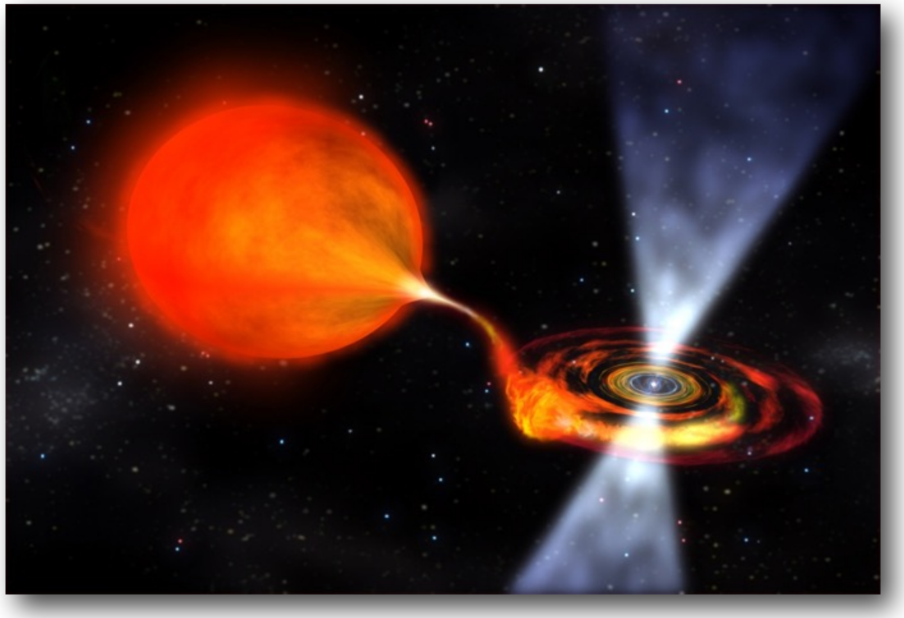
$$200\,\mathrm{Hz} \lesssim f_{\rm spin} \lesssim 600\,\mathrm{Hz}$$

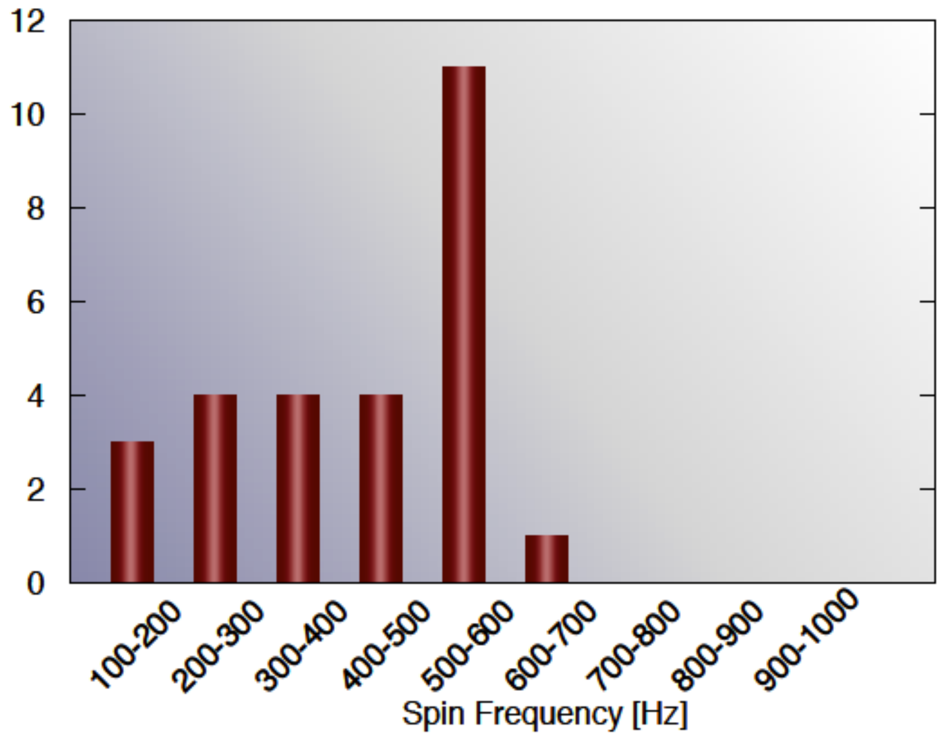
- This is well below the mass-shedding limit:

$$f_{\rm spin} \ll f_{\rm Kepler} \sim 1.5\,\mathrm{kHz}$$

- Accretion lasts $\sim 10^7 - 10^8\,\mathrm{yr}$, enough time for LMXBs to straddle the Kepler limit.

- Some process seems to halt the spin-up!

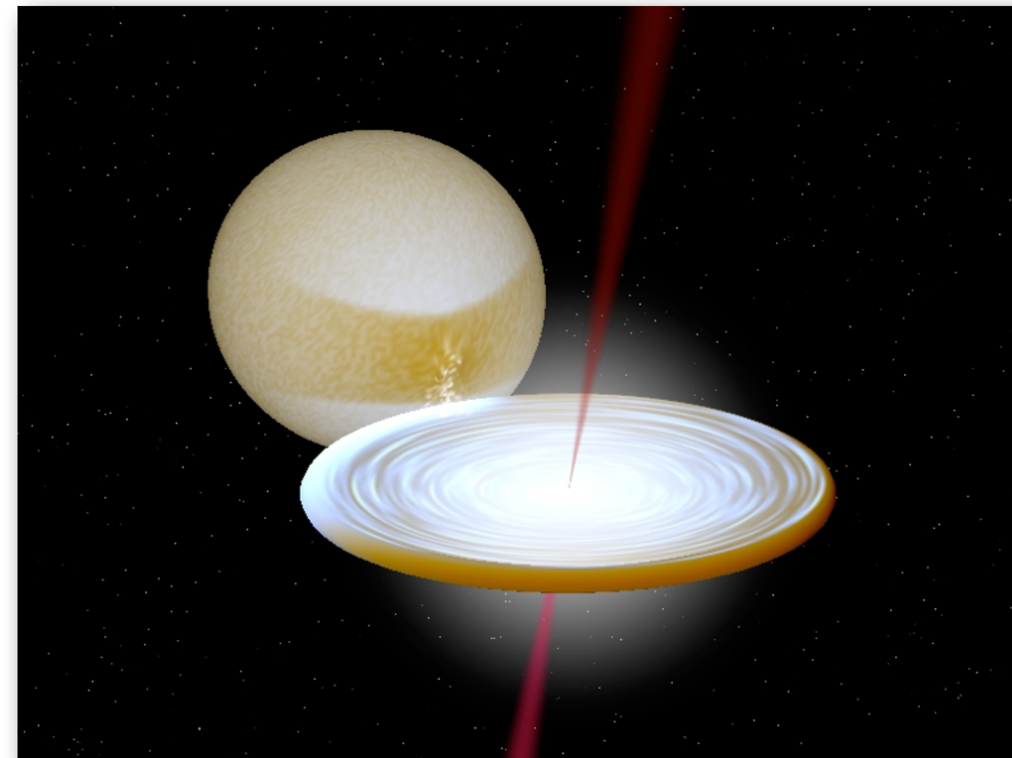




[Figure: A. Patruno]

LMXBs: halting accretion (I)

- Mechanisms for torque balance:
  - Coupling between the stellar magnetic field and the accretion disc.
  - GW torque by unstable *r*-modes

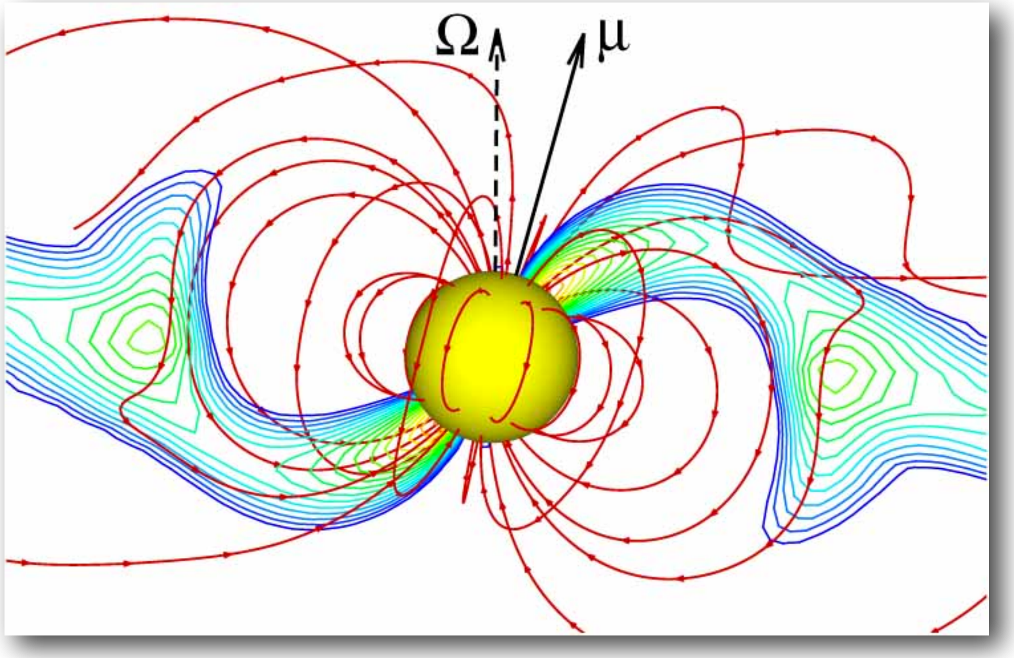


- The *r*-mode amplitude required to *balance the accretion torque:*

$$\alpha_{\rm acc} \approx 1.3 \times 10^{-7} \left(\frac{L_{\rm acc}}{10^{35}\,{\rm erg\,s^{-1}}} \right)^{1/2} \left(\frac{f_{\rm spin}}{500\,{\rm Hz}} \right)^{-7/2}$$

LMXBs: halting accretion (II)

- Magnetic disk coupling *can* provide the necessary spin-down torque (although the underpinning accretion theory is largely phenomenological).

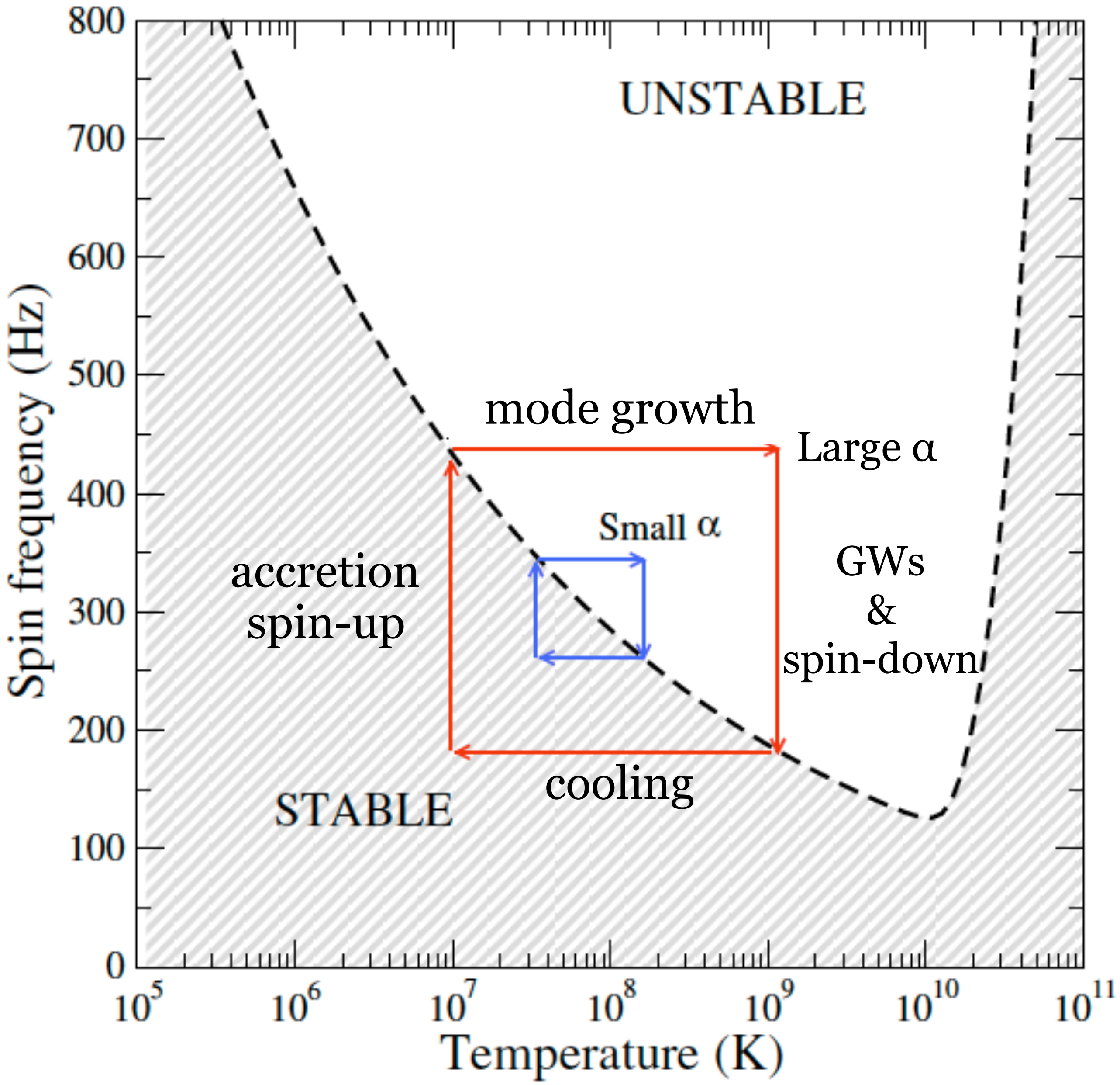


- A hint:
  the measured $\dot{f}_{\rm spin}$ of two accreting systems in quiescence [SAX J1808 & XTE J1814] is consistent with the one caused by a "canonical" surface dipole field $B \sim 10^8\,{\rm G}$.

- *r*-modes could still supply a portion of the spin-down torque.

LMXBs: Spin-temperature evolution

- The *r*-mode-driven evolution mainly depends on two factors:
  - The *T*-slope of the window at the point of entry.
  - The saturation amplitude.
- LMXBs are likely to become unstable in the negative slope portion of the instability curve.
- The figure shows the resulting thermal runaway evolution ("Levin cycle").



[Haskell et al. 2014]

r-mode cycle: GW detectability

- The detectability of r-mode-"cycling" LMXBs is a subtle issue.

- The GW duty cycle (=fraction of the cycle spent in GW emission) is :

$$D \approx \frac{t_{\rm cycle}}{10^7\,{\rm yr}} \approx \frac{10^{-11}}{\alpha^2}$$

- If α is too big, D is too low and no system would be observed being unstable.

- Combine D with the LMXB birth rate ~ $10^{-5}/{\rm yr}/{\rm galaxy}$ and lifetime ~ $10^7\,{\rm yr}$ and estimate the amplitude for which a system is always "on" in our galaxy:

$$D \lesssim 10^{-2} \;\Rightarrow\; \alpha \lesssim 10^{-4}$$

- For the system to be detectable at (say) 10 kpc we need: $\alpha \gtrsim 10^{-6}$

- *A small-ish r-mode amplitude is actually better for detecting LMXBs!*

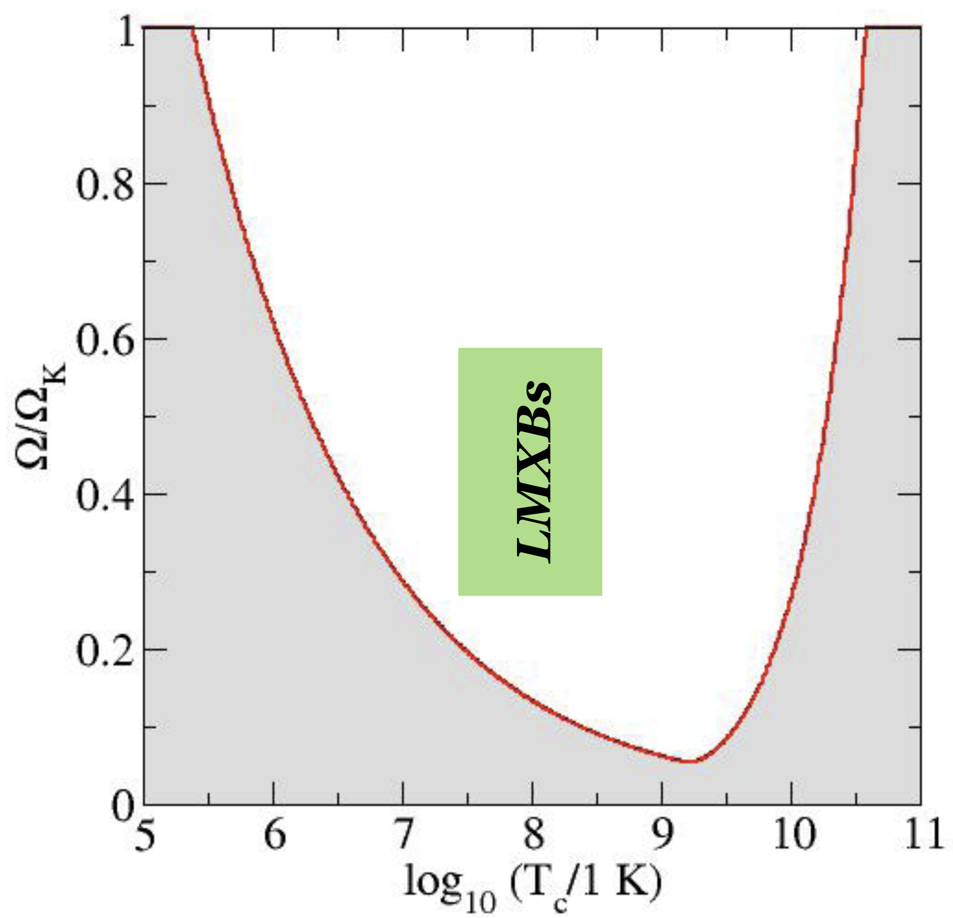
r-mode puzzle?

- Several LMXBs (and perhaps some MSPs) reside well inside the “minimal” instability window.

- These systems should experience *r*-mode-driven evolution and GW spin-down.

- This is *not* what observations suggest.

- Possible resolutions:



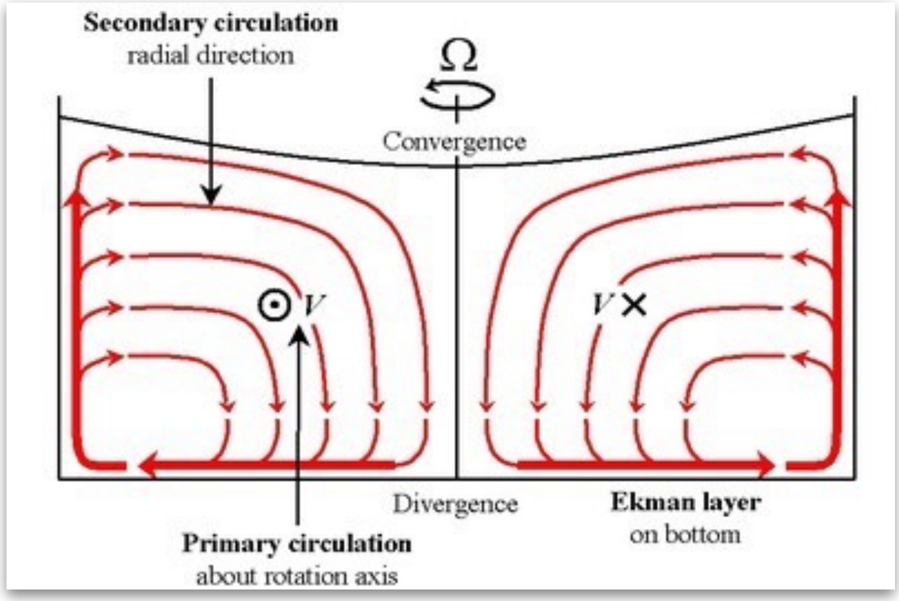
– *Additional damping*
(e.g. friction at the crust-core boundary, exotica in the core, …).

– *r*-mode amplitude *much smaller* than current theoretical predictions.

r-modes: extra damping

- Several other mechanisms could dampen the r-mode instability:

  – An Ekman-type boundary layer at the crust-core interface (i.e. the mechanism that stops tea whirling inside a cup) .



  – Bulk viscosity due to exotica (hyperons/quark matter).

  – Mutual friction due to neutron vortex- proton fluxtube interactions.

  – Coupling between the *r*-mode and superfluid modes.

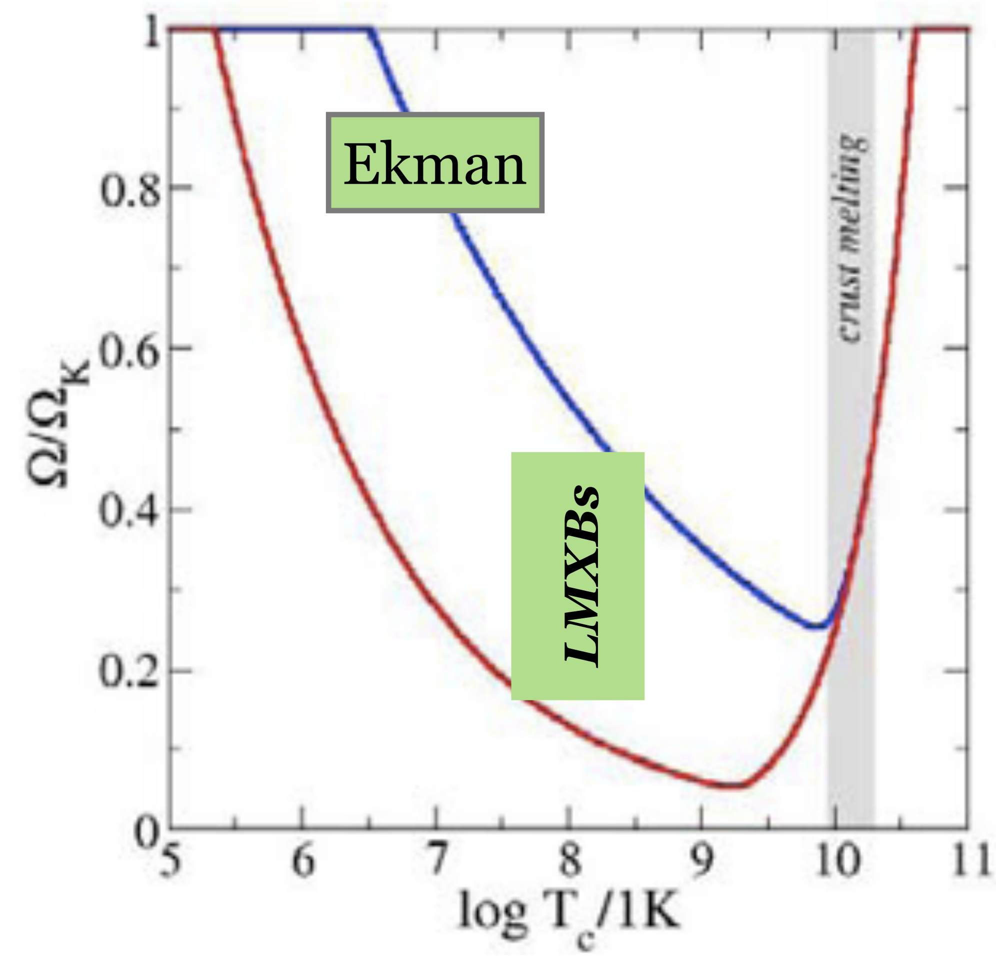
The role of the crust

- *r*-mode damping could be easily dominated by the viscous "rubbing" at the base of the crust.

- The crust is more like a jelly than solid: the resulting crust-core "slippage" reduces damping.

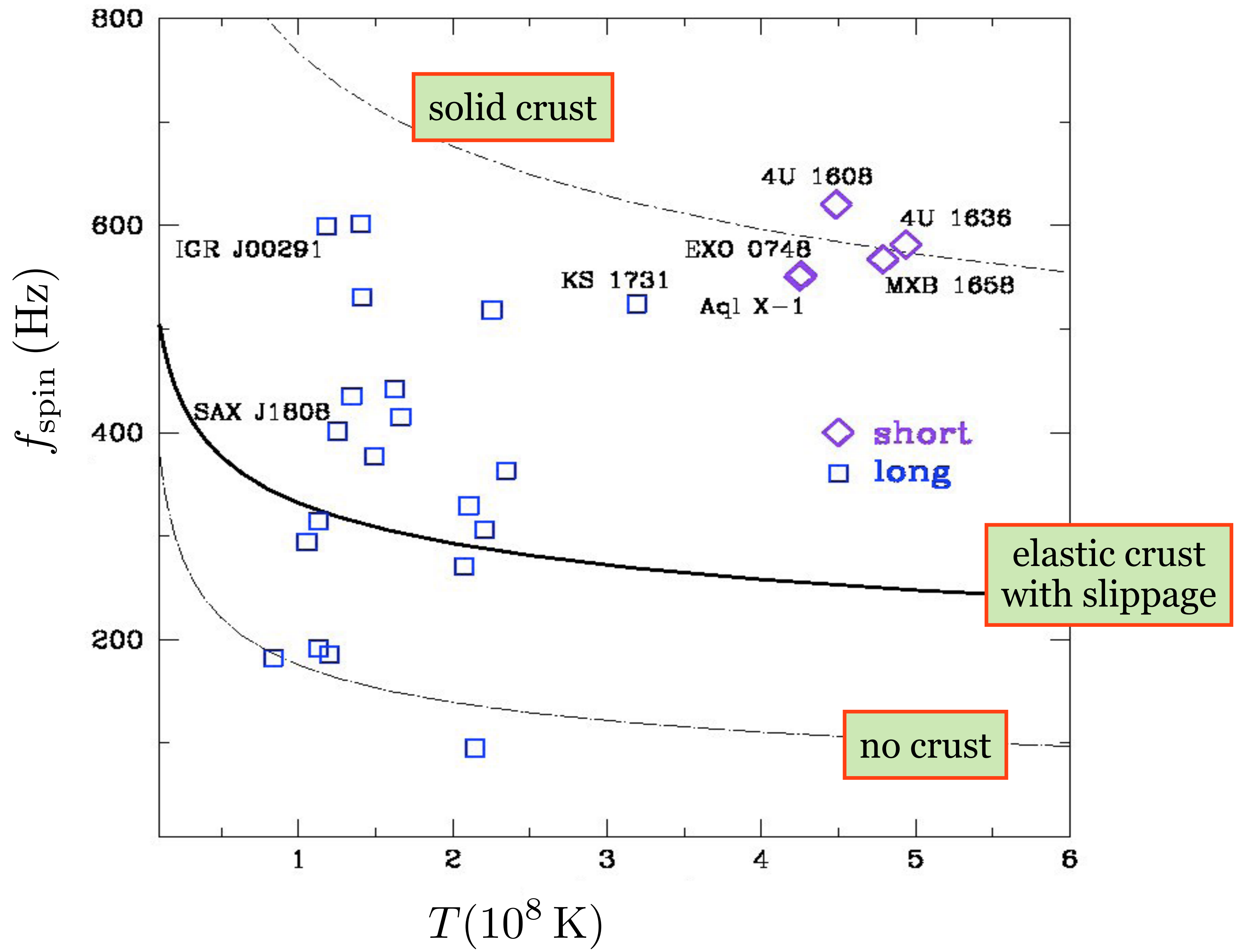
- Resonances between the *r*-mode and torsional crustal modes may also play a role.

- Existing work assumes a "sharp" crust-core transition ... but how safe is this assumption?



[KG & Andersson 2006]

r-mode window: "theory vs observations"



[Ho et al. 2011]

Magnetic boundary layer

- The Ekman layer physics is significantly modified by the local *B*-field:

– Crust-core slippage is *suppressed* (i.e. damping amplified)

– Above a threshold, the *B*-field enhances the damping rate.

– The layer's thickness grows with *B*, so *B* shouldn't be too strong.

- In LMXBs (and MSPs) the magnetic field ($B \sim 10^8$ G) can indeed lead to enhanced damping, *provided the outer core is superconducting*:

$$\frac{\dot{E}_{\rm mag}}{\dot{E}_{\rm visc}} \sim \frac{v_A}{\Omega\,\delta_E} \approx 13 \left(\frac{B}{10^8\ \rm G}\right)^{1/2} \left(\frac{T}{10^8\ \rm K}\right) \left(\frac{f_{\rm spin}}{500\ \rm Hz}\right)^{-1/2}$$

v_A = Alfven speed

δ_E =Ekman layer thickness

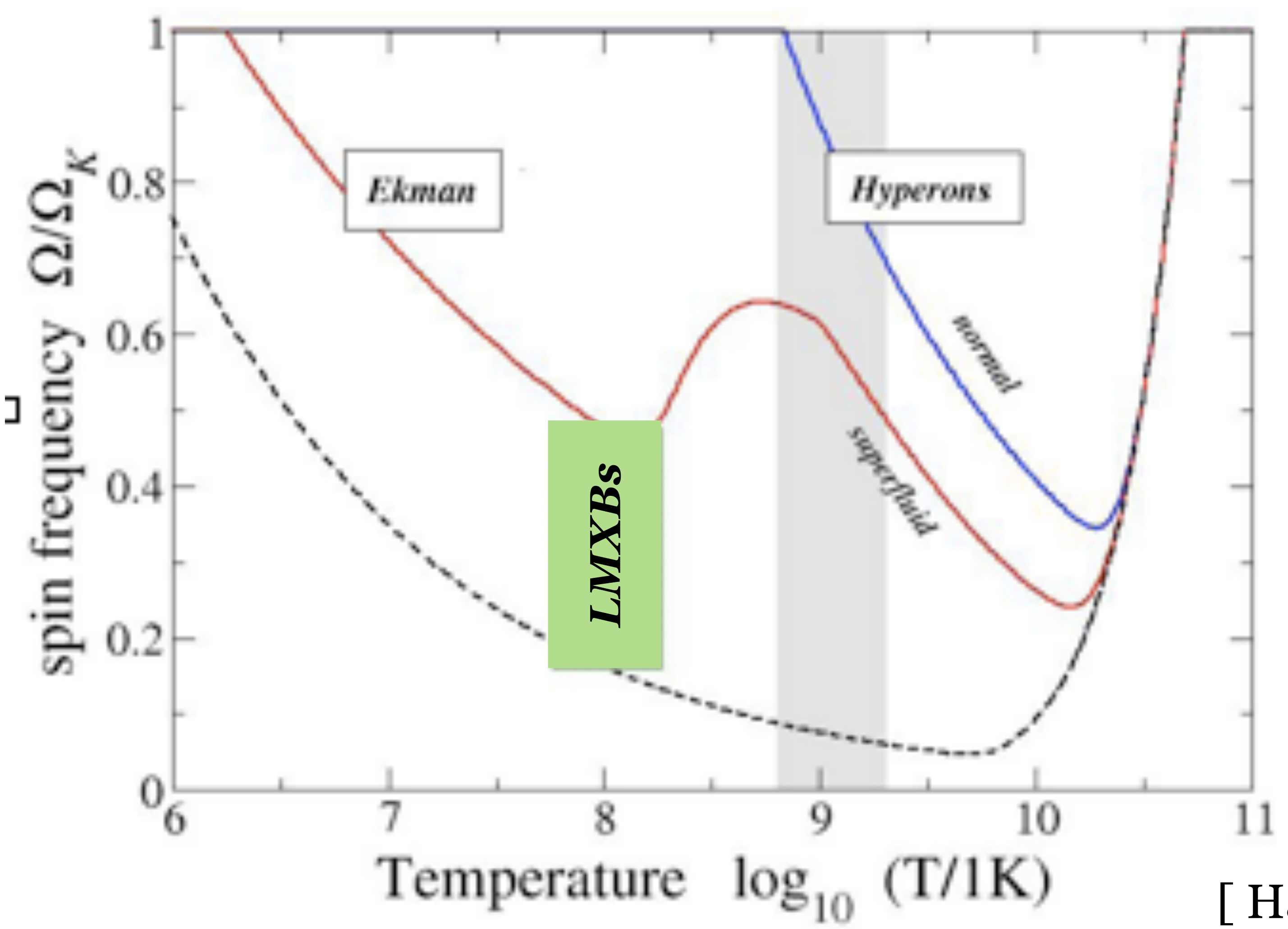
- This (approximate) result would render these systems *r*-mode-*stable*.

- But: we need more realistic crust-core boundary physics (with superfluidity/superconductivity, finite thickness transition etc.)

r-modes: exotica in the core (I)

- A neutron star core populated by hyperons and/or quarks leads to strong bulk viscosity and a *significantly* modified r-mode instability window.

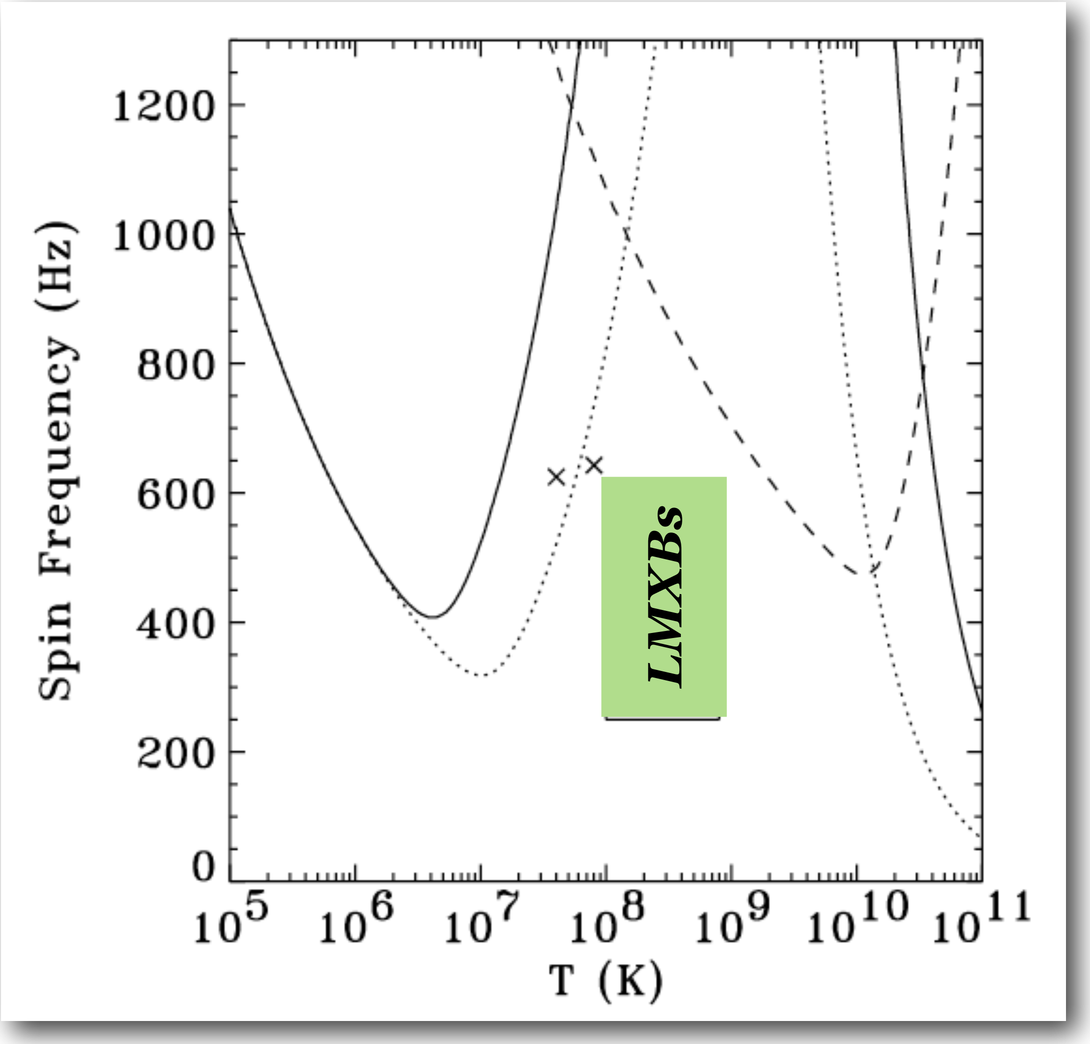
- We show representative examples of such windows (but these can vary as a function of the poorly known properties of exotic matter).



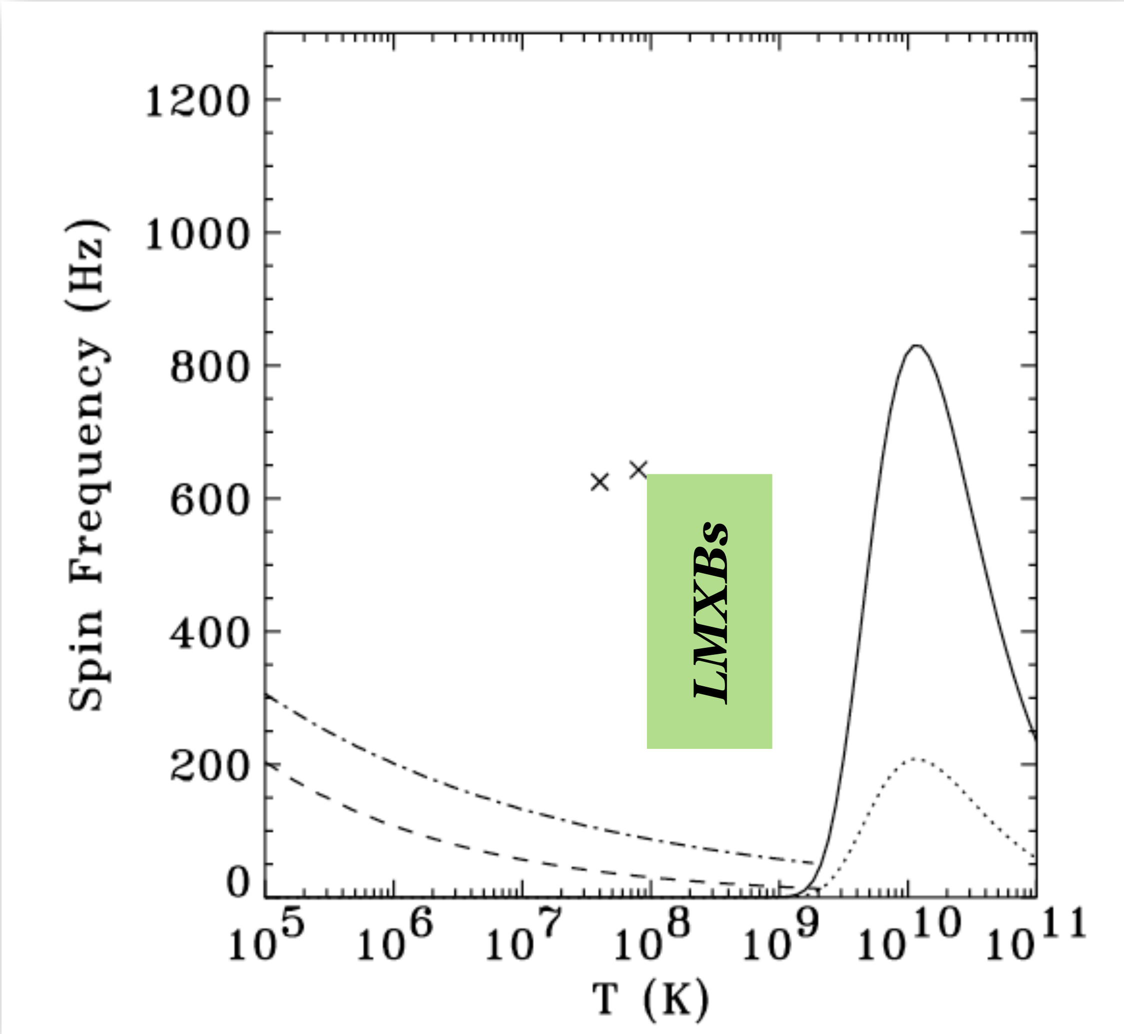
[Haskell & Andersson 2010]

r-modes: exotica in the core (II)

Quarks (without pairing)



Quarks (with pairing)

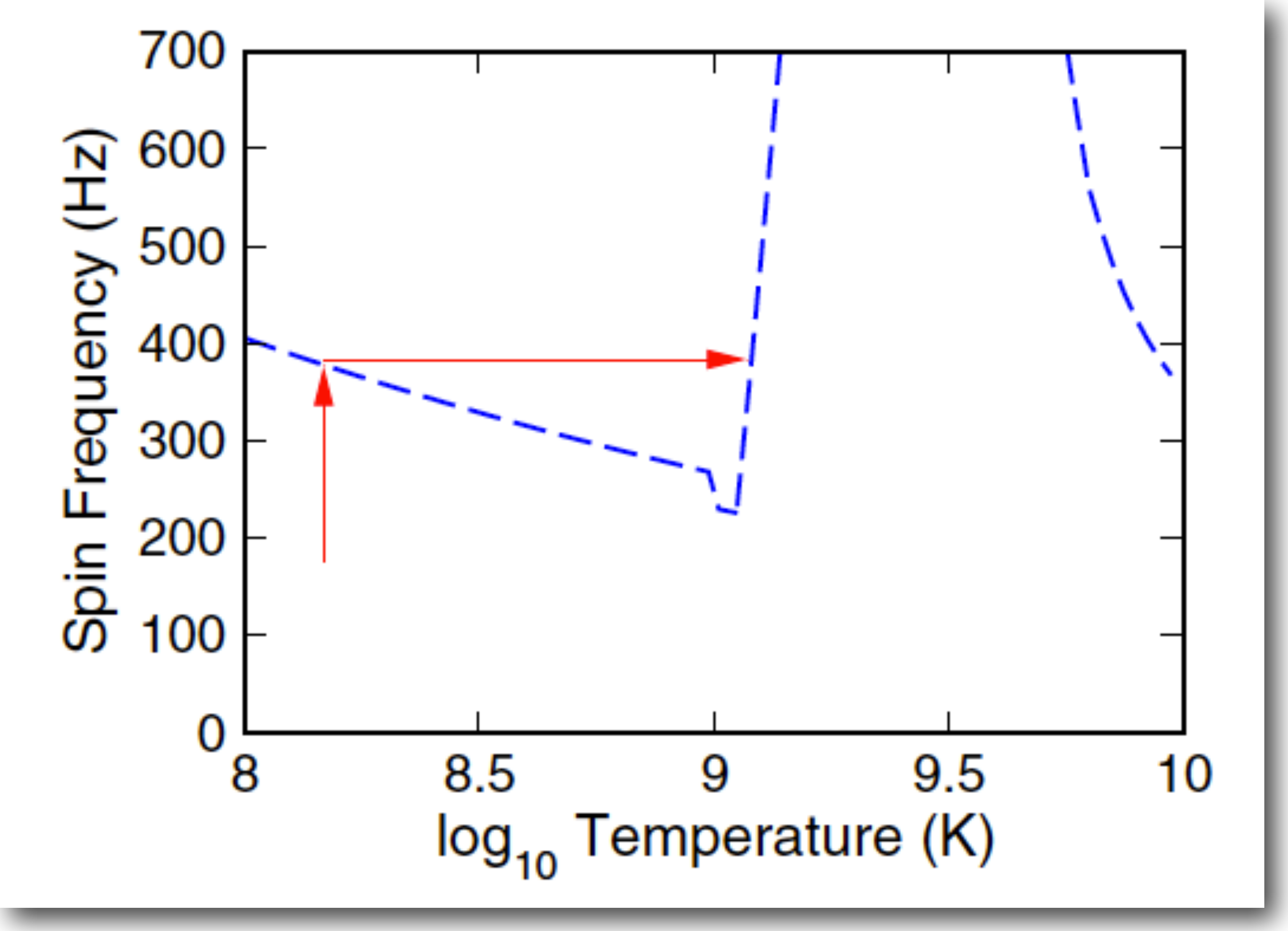


[Madsen 2000]

r-modes: persistent GW emission

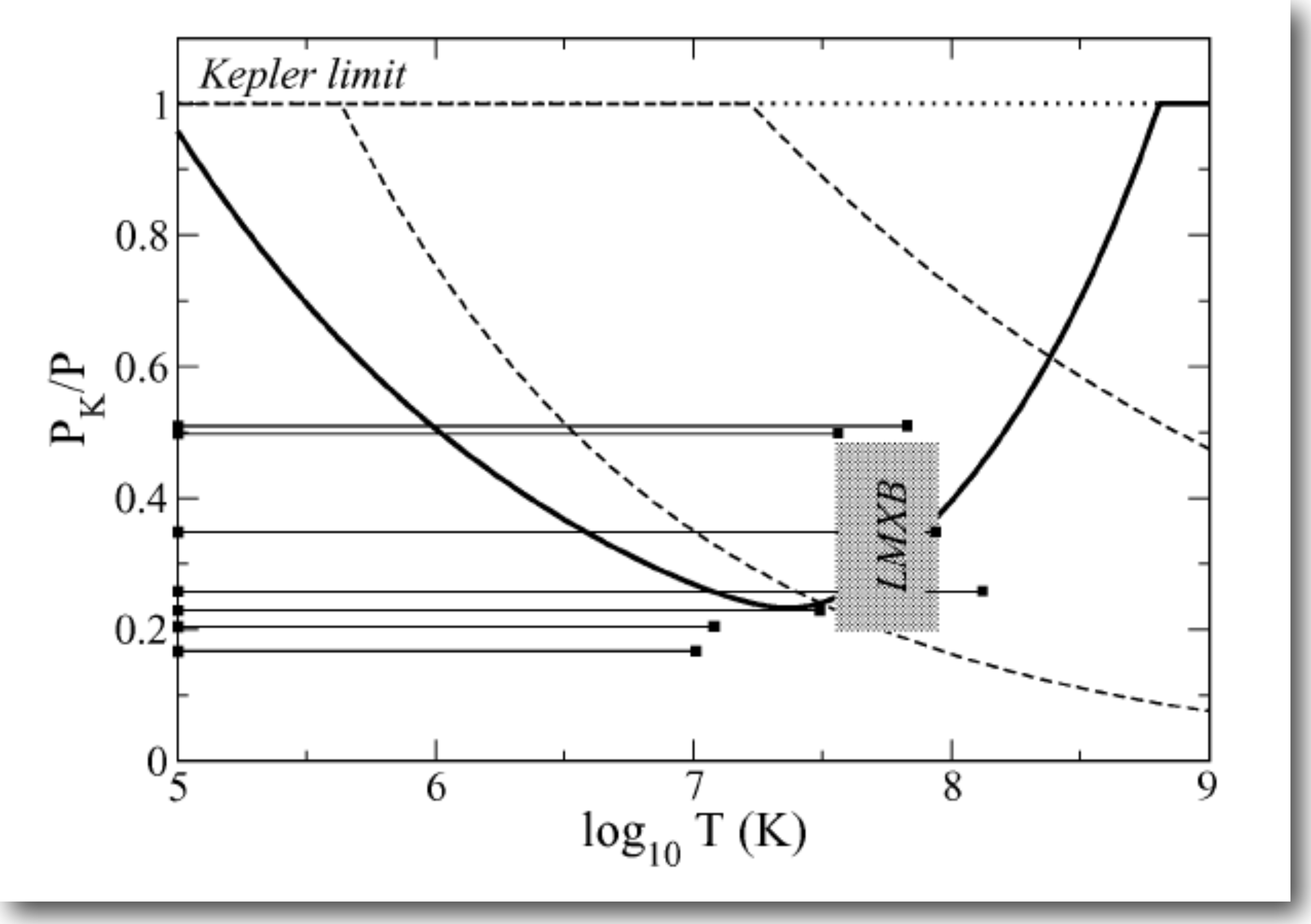
- Exotica may drive LMXB-evolution near a positive *T*-slope instability curve. Once there, the system can “hang” and become a persistent source of GWs (potentially detectable by advanced detectors).

Hyperon core



[Nayyar & Owen 2006]

Quark core



[Andersson et al. 2002]

r-mode observed? XTE 1751

- XTE 1751-305 is an AMXP (accretion-powered X-ray pulsar).

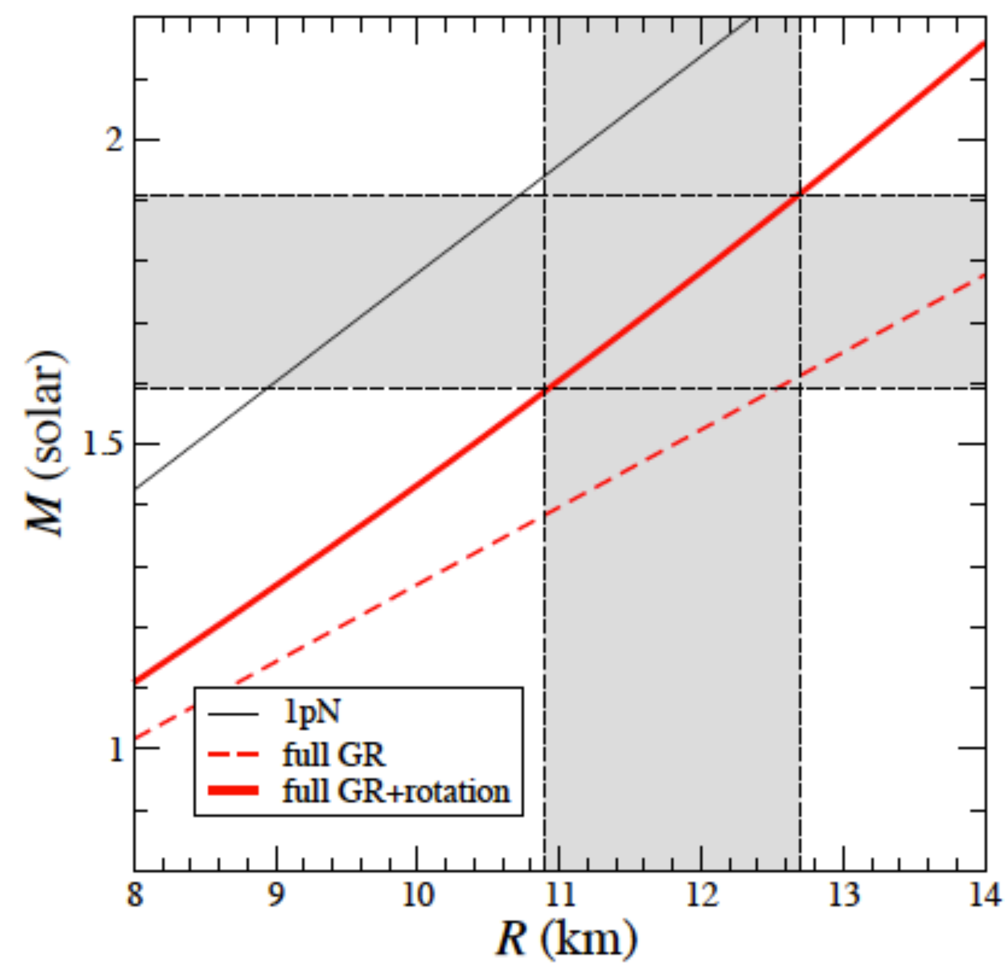
- Recently, a coherent oscillation was discovered in the light curve during a burst:

$$f_{\rm osc} = 0.572 f_{\rm spin}$$

- Provided the light curve is modulated by a global mode, the observed signal could be an *r*-mode. The numbers can match provided we account for *relativistic corrections* in the mode frequency.

- But: inferred *r*-mode amplitude too large to be reconciled with the system's spin evolution.

- Alternative interpretation:
could be a surface mode of the NS's fluid ocean.



[Andersson et al. 2014]

superfluid *r*-modes: 2-stream instability (I)

- A different kind of *r*-mode instability exists in neutron stars with superfluid neutrons.

- In the present context the *r*-mode comprises *two* flows, $\delta \mathbf{v}_{\rm n}$,$\delta \mathbf{v}_{\rm p}$ for the neutrons and protons-electrons.

- For the instability to set in, the two fluids must have a *spin lag* $\Omega_{\rm np} = \Omega_{\rm n} - \Omega_{\rm p}$ and strongly interact through the superfluid vortex array.

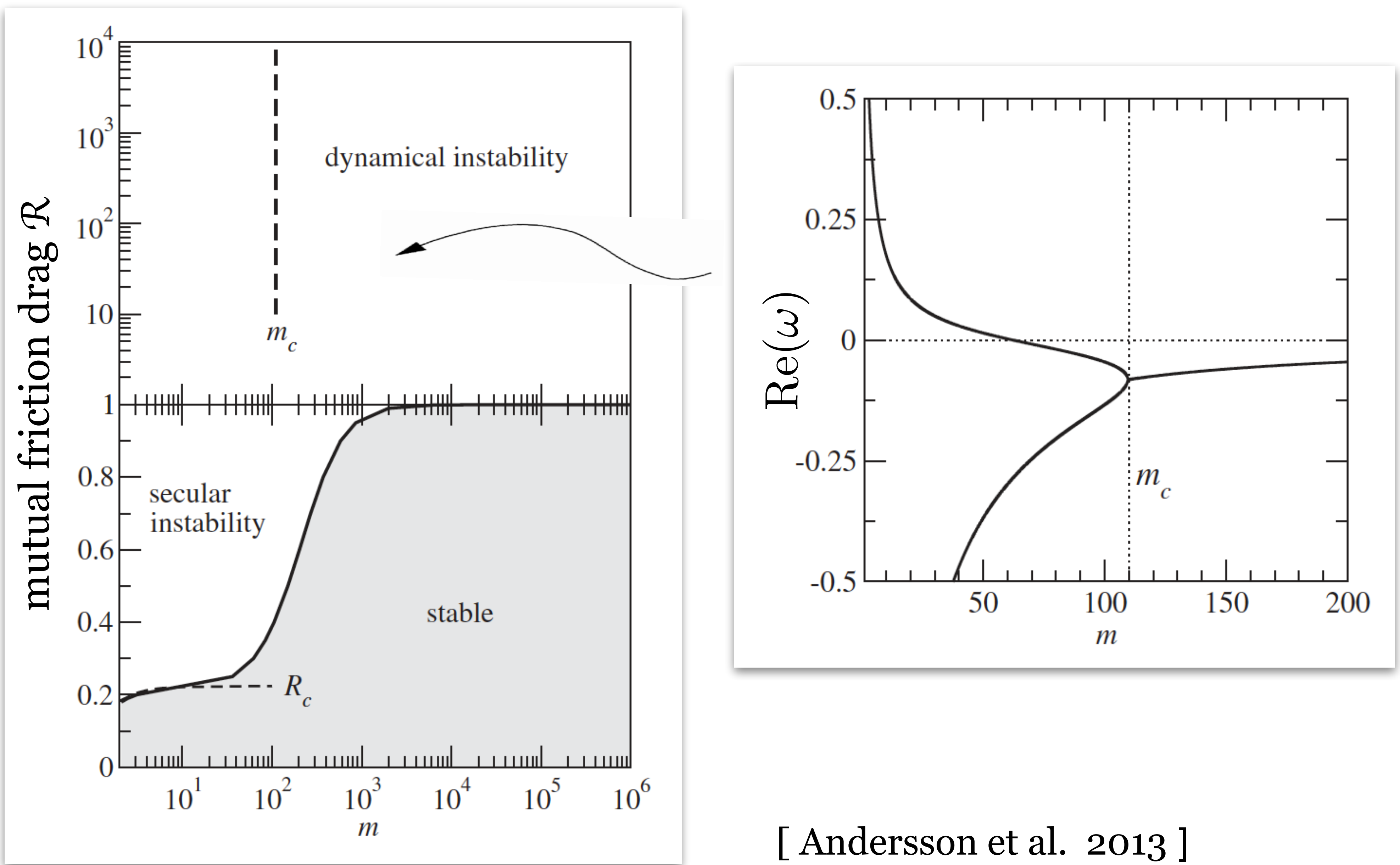
growth timescale: $\tau_{\rm grow} \approx 0.25 \left(\frac{\Omega_{\rm np}/\Omega_{\rm p}}{10^{-4}} \right)^{-1/2} \ {\rm s}$

- This is a short-wavelength instability ($\ell \gg 1$), hence irrelevant for GWs, but could be very relevant for triggering pulsar glitches!

- “2-stream” criterion: $\Omega_{\rm p} < |\omega/m| < \Omega_{\rm n}$

superfluid *r*-modes: 2-stream instability (II)

- The aforementioned instability is just a special case of a larger family which encompasses both *dynamical* and *secular* 2-stream instabilities!
- Key ingredients: two fluids with *spin-lag* and vortex *mutual friction* coupling.



[Andersson et al. 2013]

More exotic: w-mode CFS instability (I)

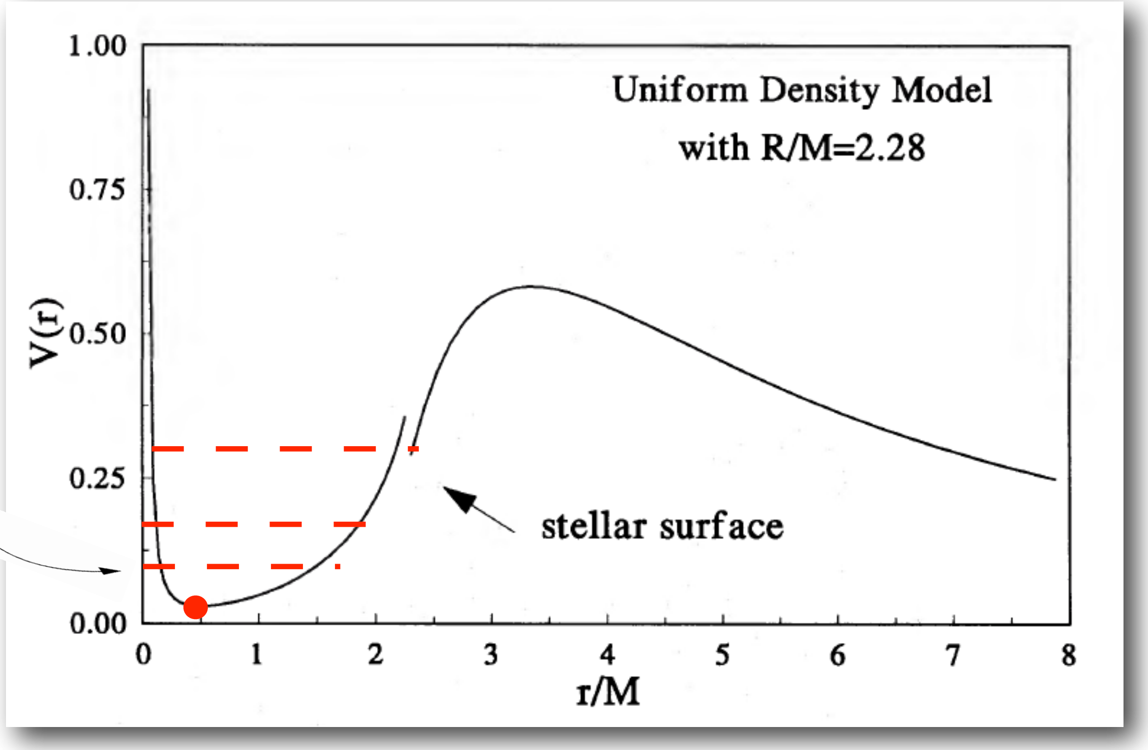
- In essence, the CFS instability is a Newtonian concept.
  The GW-driven instability is properly framed in a GR-CFS formalism.

- GR-CFS exotic possibility: w-modes (spacetime perturbations) can become unstable by spacetime frame dragging! The mode's canonical energy can become < 0 provided the star has an *ergoregion*.

- We need an ultracompact star with $R < 3M$ so that the wave potential for w-modes develops a cavity.

- This cavity harbours *long-lived* w-modes.

- We consider a slowly rotating uniform density model.



[Kokkotas 1994]

More exotic: w-mode CFS instability (II)

- Approximate calculation: use the cavity's *photon orbit* to obtain the *first w-mode* ("eikonal limit"). For simplicity, this is done for the non-rotating star.

$$\mathrm{Re}(\omega) \approx \left(\ell + \frac{1}{2}\right)\Omega_{\rm ph} \qquad M\Omega_{\rm ph} = C^{3/2}(4-9C)^{1/2}$$

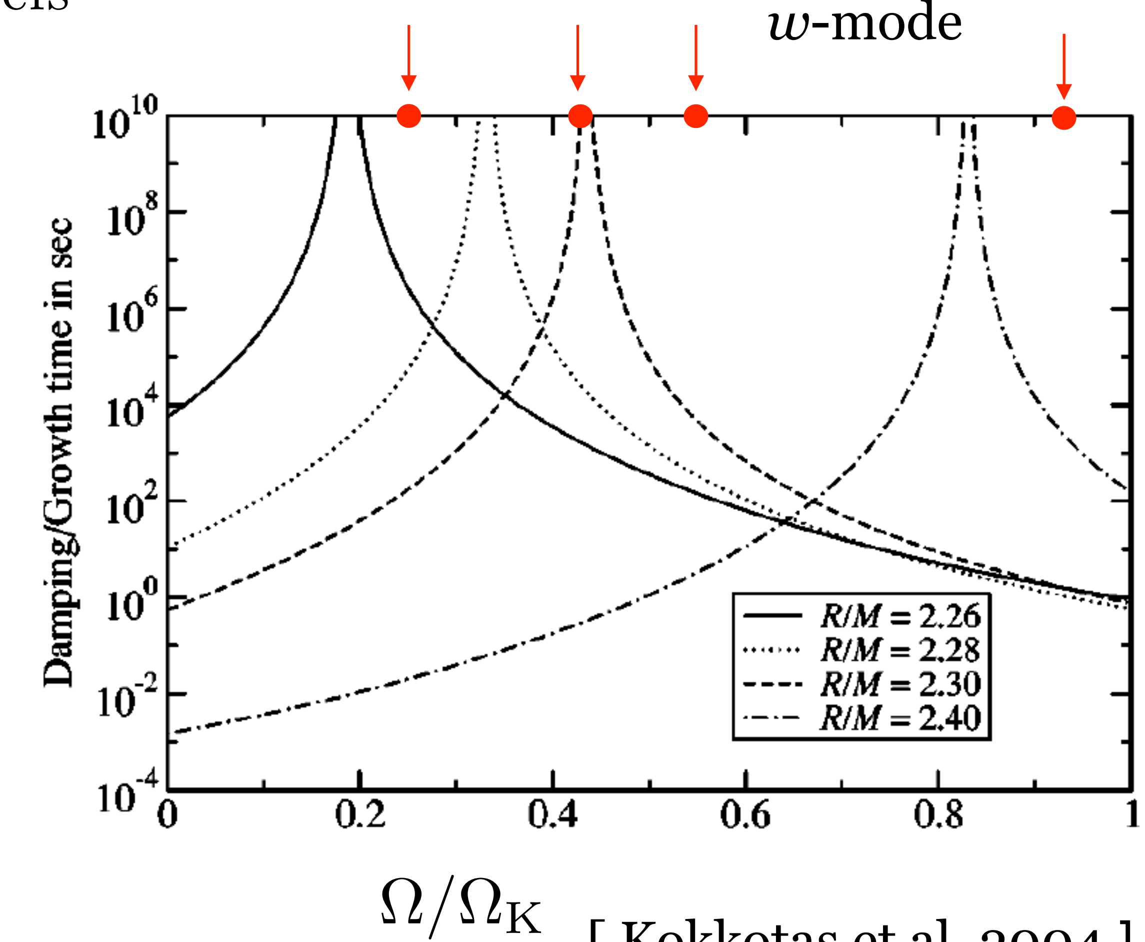
where
$C = M/R$

- The CFS instability criterion is: $\mathrm{Re}(\omega) = m\Omega_{\rm cfs}$

$$\Rightarrow \ \Omega_{\rm cfs} \approx \left(1 + \frac{1}{2m}\right)\frac{C^{3/2}}{M}(4-9C)^{1/2}$$

this approximate formula is in reasonable agreement with the exact results (figure)

first $\ell = m = 2$ w-mode



[Kokkotas et al. 2004]

- The w-mode CFS instability is nothing more but the previously known "*ergoregion instability*". It always sets in *after* an ergoregion has been formed.

Theory assignments

- Here we compile a (personally biased and far from complete) list of remaining problems relevant to the topic of this lecture.

- *r*-mode instability:

  - Detailed modelling of magnetic crust-core boundary (thickness, superfluids)
  - Resonances between the *r*-mode and other modes in superfluid NSs.
  - Full non-linear analysis of *r*-mode-driven magnetic field wind-up.
  - Lagrangian CFS theory for 2-stream *r*-mode instabilities.

- *f*-mode instability:

  - GR calculations without the Cowling approximation.
  - GR calculation of saturation amplitude.
  - "Realistic" spin-temperature evolutions for post-merger NSs.

Epilogue: a violin score

[Cover and epilogue figures courtesy of P. Pnigouras]

This is the GW “sound” of a supra-massive NS evolving through the instability window of its quadrupolar *f*-mode, starting from a Kepler limit rotation until it collapses to a BH.