Gravitational wave source modelling: astrophysical considerations

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Where are we?

- In last lecture, we discussed formulae for GW amplitudes and luminosities.
- Didn't say much about maximum or likely ellipticities or GW amplitudes.
- This will be the subject of this lecture!

Reminder of key formulae

- Definition of ellipticity:

$$\epsilon \equiv \frac{I_{xx} - I_{yy}}{I_{zz}^{\mathrm{MoI}}} = \frac{I_{yy}^{\mathrm{MoI}} - I_{xx}^{\mathrm{MoI}}}{I_{zz}^{\mathrm{MoI}}}. \quad (1)$$

- From now on, simply notation: $I_{zz}^{\mathrm{MoI}} = I$.
- GW luminosity:

$$L = \frac{32}{5}\Omega^6 (I\epsilon)^2. \quad (2)$$

- GW amplitude for a circularly polarised source:

$$h_0 \equiv \frac{4}{r}\Omega^2 I|\epsilon|. \quad (3)$$

Three sorts of “minimal assumption” upper limits

- Need to consider detailed neutron star physics to discuss maximum/likely values of ellipticity and amplitude.
- Before doing so, can use some simple **energetic arguments** to get some upper limits.
- Arguments proceed differently for different sorts of source:
    1. Targeted searches: “spin-down upper limit”.
    2. (Some sorts of) directed searches: “indirect upper limit”.
    3. All sky searches: the “Blandford argument”.
- Of course, from LIGO/Virgo non-detections, we also have *direct* upper limits!

Preamble: spin-down energetics

- Key assumption that will underpin our upper limits is *100% conversion of rotational kinetic energy into GW energy*:

$$\frac{d}{dt}\left(\frac{1}{2}I\Omega^2\right) = -\frac{32}{5}\Omega^6(I\epsilon)^2. \tag{4}$$

- This is an ODE that can be solved:

$$\Omega(t) = \frac{\Omega_0}{[1 + t/\tau]^{1/4}}, \tag{5}$$

where $\Omega_0 = \Omega(0)$ and

$$\tau = \frac{5c^5}{2^7 G}\frac{1}{I\epsilon^2\Omega_0^4}. \tag{6}$$

- If star has spun-down a lot since birth ($t/\tau \gg 1$), can approximate

$$\Omega(t) \approx \left[\frac{5c^5}{2^7 G}\frac{1}{I\epsilon^2 t}\right]^{1/4}, \tag{7}$$

independent of Ω_0.

Spin-down upper limits

- For star with known distance, spin frequency, and rate of change of spin frequency, can calculate *spin-down upper limit*.
- Simply use currently observed spin parameters in energy conservation equation ($P = 2\pi/\Omega$):

$$\epsilon_{\rm spindown} = \left[\frac{5 \dot{P} P^3}{32 (2\pi)^4 I} \right]^{1/2} . \quad (8)$$

- Substituting into formula for h_0:

$$h_0^{\rm spindown} = \sqrt{\frac{5G}{2c^3}} I^{1/2} \frac{1}{r} \left(\frac{\dot{P}}{P} \right)^{1/2} . \quad (9)$$

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Upper limits: spin down and direct

- Can plot spin down upper limit and actual ‘direct’ upper limit on the same diagram.
- Dimensionless noise curves fold-in duration of observation run, noise $\sim [S_h(f)/T_{\rm obs}]^{1/2}$ (see MAP’s lecture).
- Non-detection of Crab by LIGO in fact shows that mountain is smaller than this; current limit is $\epsilon \lesssim 9 \times 10^{-5}$.

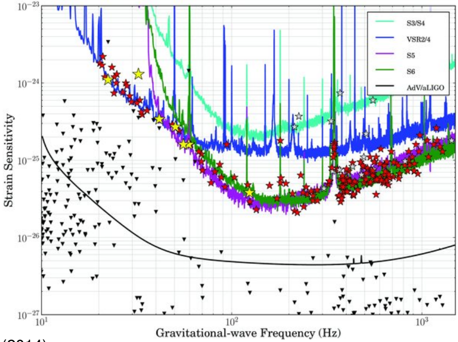

Figure: Aasi et al Ap.J. **785** 119 (2014).

Indirect upper limits

- Relevant to case where we know (or can estimate) distance and age of a source, e.g. a supernova remnant.
- Combine equation (9) for h_0 and equation (5) for $\nu(t)$:

$$h_0(t) = \frac{4G}{c^4}\frac{1}{r} I\epsilon \frac{\Omega_0^2}{\left[1+\frac{t}{\tau}\right]^{1/2}}. \tag{10}$$

- Make large spindown approximation ($t \gg \tau$) and use definition of τ:

$$h_0(t) \approx \left(\frac{5G}{8c^3}\right)^{1/2} I^{1/2} \frac{1}{r}\frac{1}{t^{1/2}}, \tag{11}$$

independent of ν_0 and ϵ !

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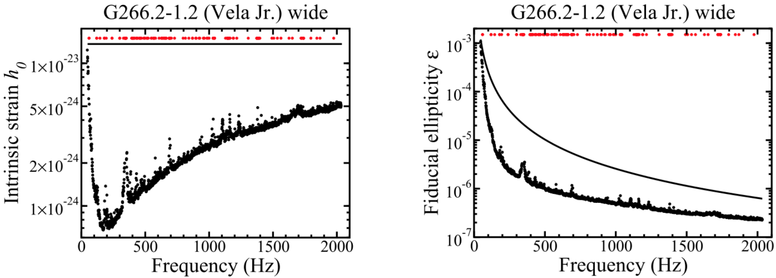
Indirect limits cont...

- This sort of analysis was used to select a set of 9 supernova remnants where LIGO could beat the indirect limit. See Aasi et al. (2015).
- Specimen result: upper limits on h_0 and ϵ for "Vela Junior":



All-sky searches: the “Blandford argument”

- Now assume no particular knowledge of source.
- Assume only that there exists a population of (unseen!) neutron stars, born spinning fast, that spin down only through GW emission.
- Sometimes known as “gravitar”!
- Following argument given briefly in Thorne’s article in “300 Years of Gravitation”, and attributed to Blandford.
- Model Galaxy as flat disk, radius $R_{\rm G}$.
- Assume stars born ever Δt years.
- Let N denote the number of such stars with age T or less:

$$N \approx \frac{T}{\Delta t}. \qquad (12)$$

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"Blandford" cont...

- Let d denote average nearest-neighbour separation of this set of stars, so area per star is:

$$d^2 \approx \frac{\pi R_{\mathrm{G}}^2}{N} \Rightarrow d \approx R_{\mathrm{G}} \sqrt{\frac{\pi}{N}}. \tag{13}$$

- Substitute using $N = T/\Delta t$:

$$d = R_{\mathrm{G}} \sqrt{\frac{\pi \Delta t}{T}}. \tag{14}$$

- Recall equation (11) for $h(t)$, which assumed stars have spun down significantly since birth:

$$h_0(t) \approx \left(\frac{5G}{8c^3}\right)^{1/2} I^{1/2} \frac{1}{r} \frac{1}{t^{1/2}}. \tag{15}$$

- Identify $t \sim T$ and inset above estimate of distance (from Earth):

$$h_0 \approx \left(\frac{5G}{8\pi c^3}\right)^{1/2} I^{1/2} \frac{1}{R_{\mathrm{G}}} \frac{1}{\Delta t^{1/2}}. \tag{16}$$

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“Blandford” cont. . .

- Slightly more careful treatment in Abbott et al. (2007):

$$h_0 \approx \left[\frac{5GI}{c^3 \Delta t\, R_{\rm G}^2} \ln \left(\frac{f_{\rm max}^{\rm GW}}{f_{\rm min}^{\rm GW}} \right) \right]^{1/2} . \tag{17}$$

- Parameterising in terms of a rather optimistic estimate of the interval between Galactic supernova:

$$h_0 \approx 4 \times 10^{-24} \left(\frac{30 \text{ years}}{\Delta t} \right)^{1/2} . \tag{18}$$

- However, numerical modelling by Knispel & Allen (2008) obtains lower maximum amplitudes, by an order of magnitude.
- Original assumptions of flat disk and steady-state populations not fulfilled for realistic population models.

Maximum amplitudes/ellipticities from neutron star models

- GW amplitude given by:

$$h_0 = 1.05 \times 10^{-27} \left(\frac{\epsilon}{10^{-6}} \right) \left(\frac{f_{\rm GW}}{100\,{\rm Hz}} \right)^2 \left(\frac{10\,{\rm kpc}}{r} \right). \quad (19)$$

- Have looked at upper limits using simple energetics/population arguments.
- Turn now to consideration of actual neutron star structure to gain insight into maximum ellipticities.
- Two possible types of deformation:
  1. Strains in solid crust, or possibly core, or
  2. Magnetic forces.
- Again, some simple arguments can help. Start with elastic strains.

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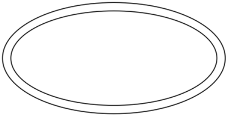
Triaxial neutron stars: Crustal strains

To get a feel for the maximum possible mountain size, we can use a simple thought experiment:

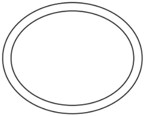
'Hand of God'

oblateness ϵ_0



Relaxed (i.e. physical configuration)

oblateness $\epsilon < \epsilon_0$



Triaxial neutron stars: Crustal strains

- The energy of the star can be written as:

$$E(\epsilon) = E_{\rm spherical} + A\epsilon^2 + B(\epsilon - \epsilon_0)^2. \quad (20)$$

- Actual shape from $\partial E/\partial\epsilon = 0$

$$\Rightarrow \epsilon = \frac{B}{A+B}\epsilon_0. \quad (21)$$

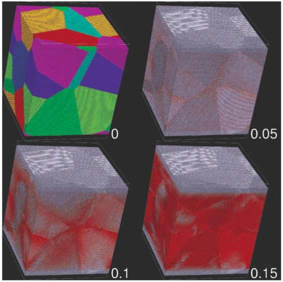
- $B \sim$ Coulomb binding energy of solid phase $\sim \mu V_{\rm solid}$.
- $A \sim$ gravitational energy of *whole* star $\sim GM^2/R \sim 10^{52}$ ergs.
- In all plausible cases, $B \ll A$.
- Crustal strain $\sim |\epsilon - \epsilon_0|$ can be no larger than the breaking strain $u_{\rm break}$ of crust.

Elastic mountains: 'normal' neutron stars

- Maximum elastic mountain size determined by balance between gravitational and elastic forces:

$$\epsilon \approx \frac{\mu V_{\rm crust}}{GM^2/R} \times u_{\rm break} \approx 10^{-6} \left(\frac{u_{\rm break}}{10^{-1}} \right).$$

- Shear modulus has long been known to be $\lesssim 10^{29}$ erg cm^{-3}.
- Large-scale molecular dynamics of Horowitz & Kadau (2009) indicate very high breaking strain, $\theta_{\rm max} \sim 0.1$ (see Figure), for some parts of crust at least.
- Plastic flow may relax crust on longer timescales (Chugunov & Horowitz 2010).



Elastic mountains: doing it properly

- Need to solve coupled equations:
  1. Equation of force balance, including "Hooke's law":

$$0 = -\rho \nabla_a \Phi - \nabla_a p + \nabla^b t_{ab}, \tag{22}$$

     where elastic stress tensor is:

$$t_{ab} = \mu \left(\nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{3} \delta_{ab} \nabla^c \xi_c \right) . \tag{23}$$

  2. Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho. \tag{24}$$

  3. Equation of state

$$P = P(\rho). \tag{25}$$

- Normally solve perturbed form of these, with respect to a spherical fluid zero strain background; see e.g. Ushomirsky, Cutler & Bildsten (2000).

Elastic mountains: more exotic scenarios

- Exotic states of matter *might* lead to solid cores giving larger maximum allowed ellipticites.
- $\epsilon_{\rm max} \sim 10^{-1}$ possible for solid quark stars, 10^{-3} for hybrid stars (Johnson-McDaniel & Owen 2013).
- Crystalline colour superconducting quark matter also relevant (Mannarelli et al 2007) leading to similarly large maximum ellipticities (Haskell et al 2007 and Lin 2007).
- Lack of detection of such a large mountain *does not* rule out such exotic states of matter ...
- ... need estimates of *likely* ellipticities, not just upper bounds!

Mountain building

How might such an elastic mountain by formed in the first place?

- Bildsten (1998) investigated temperature/composition asymmetries.
- Viability of mechanism confirmed by Ushomirsky, Cutler & Bildsten (2000).
- But what creates temperature asymmetry? Open question!

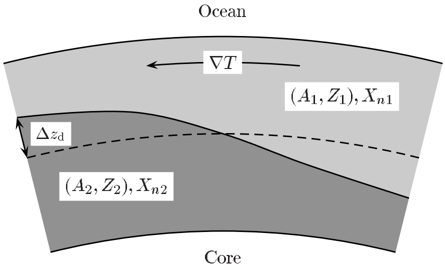


Figure: Ushomirsky, Cutler & Bildsten (2000)

Magnetic mountains: simple estimates

- Magnetic field lines have an effective tension, and deform star (Chandrasekhar & Fermi 1953). Roughly,

$$\epsilon \sim \frac{\int B^2 \, dV}{GM^2/R} \sim 10^{-12} \left(\frac{B}{10^{12}\,\mathrm{G}} \right)^2 .$$

- If protons form type II superconductor, magnetic field confirmed to fluxtubes. Effect of this is to increase tension by a factor of H_c/B, where $H_c \sim 10^{15}$ G, increasing ellipticity:

$$\epsilon \sim 10^{-9} \frac{B}{10^{12}\,\mathrm{G}} .$$

- Either way, ellipticities are small, GWs undetectable.

Magnetic mountains: doing it properly (Newtonian)

- Need to solve coupled equations:
  1. Force balance:
  $$0 = -\nabla P - \rho\nabla\Phi + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{26}$$
  2. Poisson equation:
  $$\nabla^2\Phi = 4\pi G\rho. \tag{27}$$
  3. Equation-of-state equations:
  $$P = P(\rho). \tag{28}$$
  4. "Solenoidal constraint":
  $$\nabla \cdot \mathbf{B} = 0. \tag{29}$$
- Some calculations done perturbatively about spherical unmagnetised background (e.g. Haskell et al. (2008)), some fully non-linearly (e.g. Lander & Jones (2009)).
- However, almost all equilibrium solutions in literature prove to be dynamically unstable (e.g. Lander & Jones 2012); ongoing area of research!
- Full GR solutions have been obtained too (Bonazzloa & Gourgoulhon (1996)).

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'Exotic' magnetic mountains

- If CFL or 2SC phases occur in neutron star cores, can get *colour-magnetic flux tubes* (Iida & Baym 2002, Iida 2005, Alford & Sedrakian 2010).
- This leads to flux tube tension $\sim 10^3$ larger than in protonic superconductivity case. Glampedakis, DIJ & Samuelsson (2012) estimate ellipticity:

$$\epsilon_{\rm CFL} \sim 10^{-7} \left(\frac{f_{\rm vol}}{1/2}\right) \left(\frac{B_{\rm int}}{10^{12}\,{\rm G}}\right) \left(\frac{\mu_{\rm q}}{400\,{\rm MeV}}\right)^2,$$

  where

  - $f_{\rm vol}$ = fraction of stellar volume in deconfined state,
  - $B_{\rm int}$ = *internal* magnetic field strength,
  - $\mu_{\rm q}$ = quark chemical potential.

- Can allow for internal field to be some multiple of external field:

$$B_{\rm int} = \alpha B_{\rm ext}.$$

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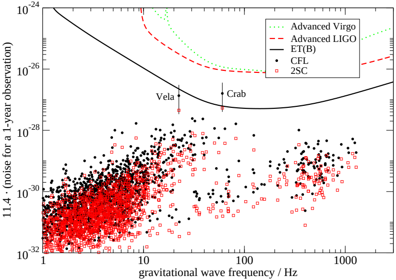
'Exotic' magnetic mountains cont . . .

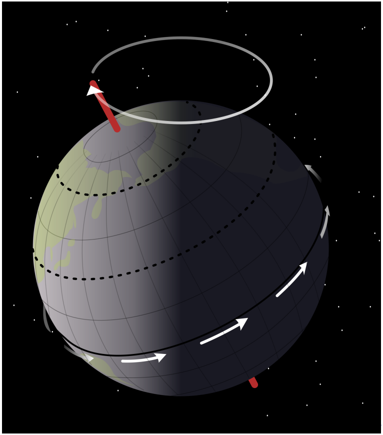
- For given stellar parameters $f_{\rm vol}$, α and $\mu_{\rm q}$ can then balance observed spin-down of pulsars against combined GW & EM torque to estimate $B_{\rm int}$ and hence h.
- GW amplitudes scale as $h \sim f_{\rm vol}\alpha\mu_{\rm q}^2$; for sensible values ($f_{\rm vol} = 0.5$, $\alpha = 2$, $\mu_{\rm q} = 400$ MeV) obtain:



Clearly of interest for Crab and Vela pulsars.

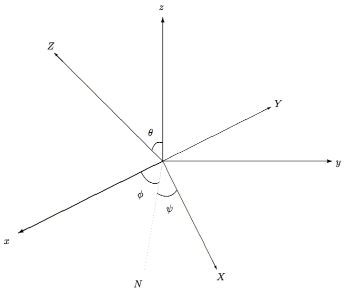
Free precession

- Have so far considered steady rotation about a fixed axis.
- More generally, rotating body can undergo “free precession”.
- Earth has a 14 month free precession period, known as the “Chandler wobble”.
- Free precession affects both GW and electromagnetic (particularly radio pulsar) emission.
- It also depends sensitively on the star’s internal structure.



What is free precession?

- Best described in terms of Euler angles giving the body's orientation with respect to inertial frame:



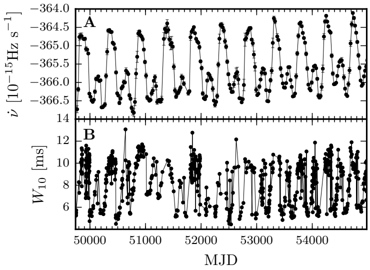
- Will specialise to case of *biaxial body*, moments of inertia (I_x, I_x, I_z), and assume body close to spherical.
- Motion then a superposition of two rotations:
  1. Symmetry axis Oz rotation rapidly about fixed angular momentum vector, in cone of half-angle θ, the "wobble angle", at rate $\dot{\phi}$.
  2. Star also rotates slowly about symmetry axis, at "precession frequency", $\dot{\psi} \approx -\dot{\phi}(I_z - I_x)/I_z$ (assuming small θ).

Gravitational wave emission

- To calculate GW emission, follow procedure described in last lecture, but with more complex rotation matrix to generate quadrupole moment tensor with respect to inertial frame:
$$R_{ab} = [R^z(\phi) R^x(\theta) R^z(\psi)]_{ab}. \tag{30}$$
- Exactly the same procedure as in lecture 1 then leads to the GW field.
- Find emission at *two* frequencies: $\dot{\phi}$ and $2\dot{\phi}$.
- GW amplitude at each proportional to $\dot{\phi}^2 (I_z - I_x)$.
- $2\dot{\phi}$ harmonic emitted preferentially along angular momentum axis.
- $\dot{\phi}$ harmonic emitted preferentially in plane perpendicular to angular momentum axis.
- No GW emission related to $\dot{\psi}$ rotation about symmetry axis.

Electromagnetic emission

- Effect of precession on EM emission more complex.
- Unless radio pulsar beam happens to lie along symmetry axis, will get slow modulations in spin-down rate on the (long) free precession timescale.
- Also, observer's "cut" through the beam will be modulated.
- Possible that this has already been observed; best candidate is PSR B1828-11:



Dual-harmonic searches?

- Almost all LIGO/Virgo continuous wave searches have assumed GW emission only at 2ν (not ν).
- Why?
- Free precession does *not* seem to be common in pulsars, and . . .
- . . . the best precession candidates are all slowly spinning (too slow for LIGO/Virgo).
- Exceptions are a few “narrow band” searches, where narrow frequency band around 2ν searched.
- But the situation may be more complicated . . .

GW emission: effect of a pinned superfluid

- Part of interior superfluid can “pin” to rest of star:

$$J_a = I^{\mathrm{C}}_{ab}\Omega^{\mathrm{C}}_b + I^{\mathrm{SF}}\Omega^{\mathrm{SF}}_a. \tag{31}$$

- Pinned superfluid acts as a gyroscope, sewn into the star!
- A steadily rotating star can then rotate about an arbitrary axis, giving GW emission at both ν and 2ν (DIJ 2010).
- Implication:

Observation of GW at both f and 2f from a steadily spinning star will provide evidence for pinned superfluidity within the star.

- Such a search has been carried out on “old” S5 data (Pitkin et al 2015); didn’t find anything!

Summary

- Can use simple energetics arguments to estimate upper limits for GW emission for targeted, directed, and all-sky searches.
- Have already entered the regime where detection energetically possible for targeted and directed searches.
- Need to look at detailed neutron star physics to decide if such upper limits are compatible with known equation of state/neutron star structure, let alone if they are realistic.
- Need to think more about relaxing assumption of GW emission at *exactly twice* the spin frequency, in case we are missing something important.
- Theoretical uncertanties are sufficiently large that one cannot confidently predict what sort of detector sensitivity will lead ot first continuous wave detections, but. . .
- . . . pay-off of a detection, in terms of learning about stellar interior, will be enormous, providing one can solve the 'inverse problem'!

Exercises

1. Verify that equation (5) follows from the energy balance equation that precedes it.
2. Use the online pulsar data archive

   http://www.atnf.csiro.au/research/pulsar/psrcat/

   to compute GW upper limits on h and ϵ for some of the known pulsars.
3. The argument relating to stellar deformations can be generalised by including rotation in equation (20) to give:

$$E(\epsilon) = E_{\text{spherical}} + A\epsilon^2 + B(\epsilon - \epsilon_0)^2 + \frac{J^2}{2I_0(1+\epsilon)}, \tag{32}$$

   where J is the angular momentum, and $I_0(1+\epsilon)$ the moment of inertia. By minimising E with respect to ϵ at fixed J, show that the ellipticity ϵ is the sum of *two* pieces, one (proportional to the square of the rotation rate) supported by the rotation, and the other supported by crustal strain. In the case where the assumed axisymmetry is broken, it is this second piece that sets the level of the gravitational wave emission, i.e. it is the bit relevant for GW emission from mountains, and from free precession.