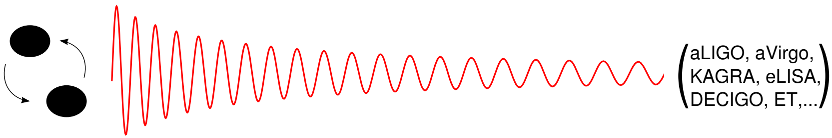
Analytical relativity modelling of coalescing compact binaries

Alexandre Le Tiec

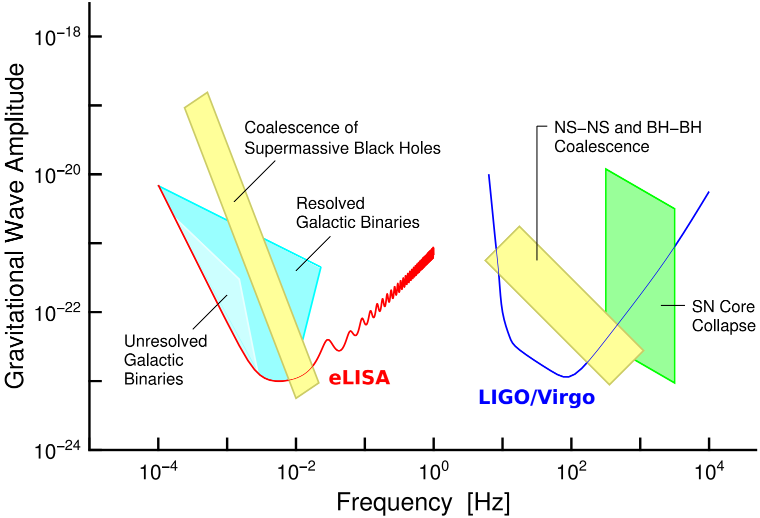
Laboratoire Univers et Théories
Observatoire de Paris / CNRS



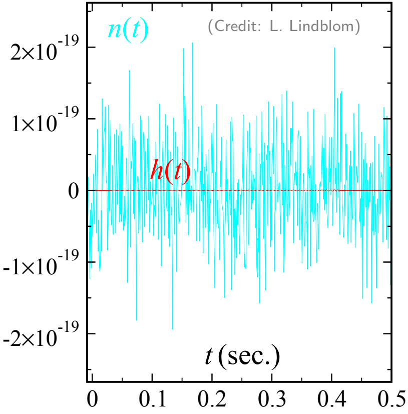
Outline

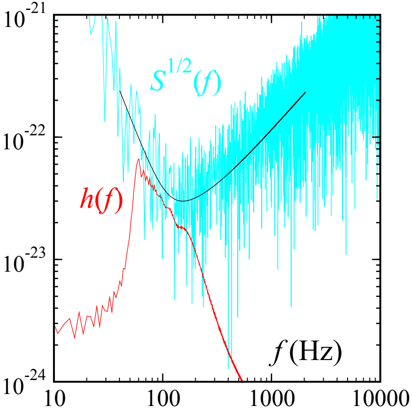
Outline

Main sources of gravitational waves

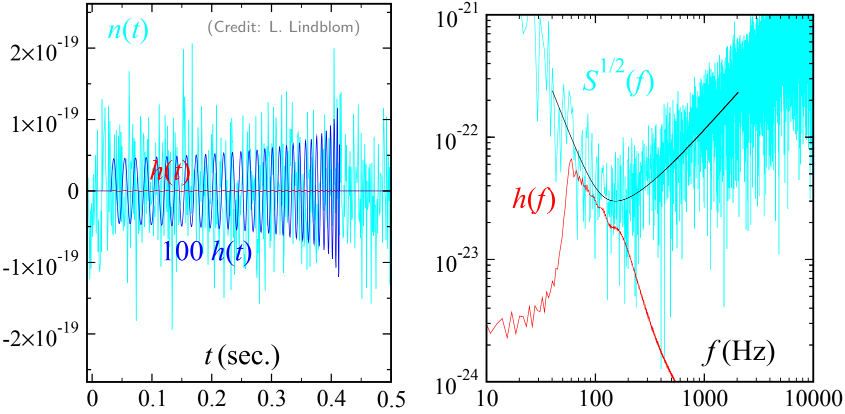


Need for accurate template waveforms

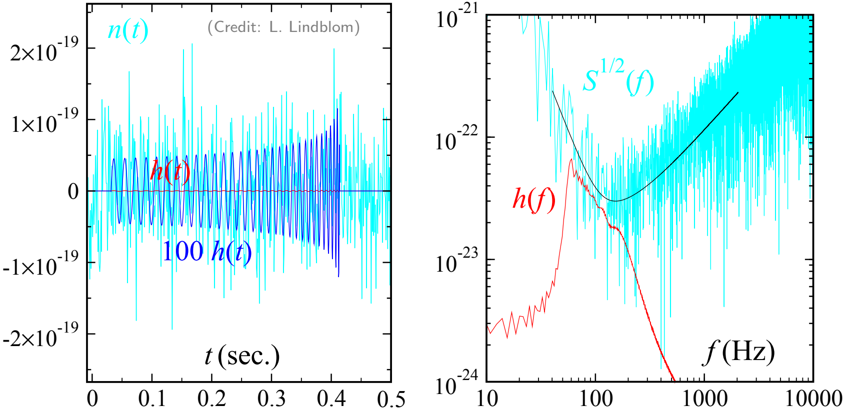




Need for accurate template waveforms

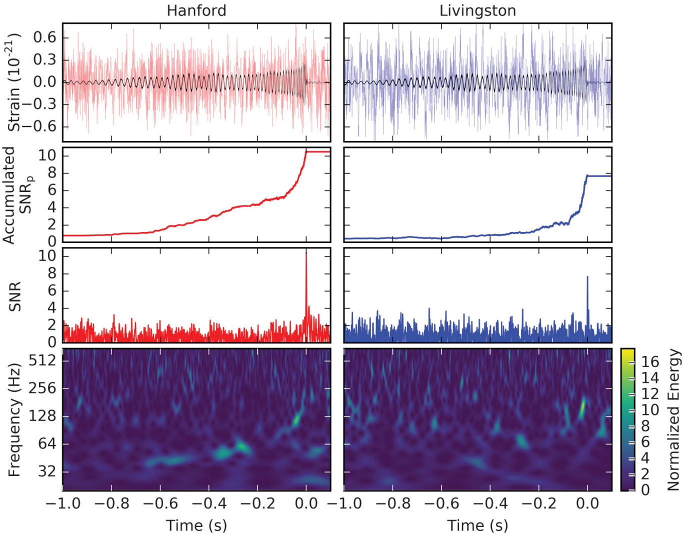


Need for accurate template waveforms



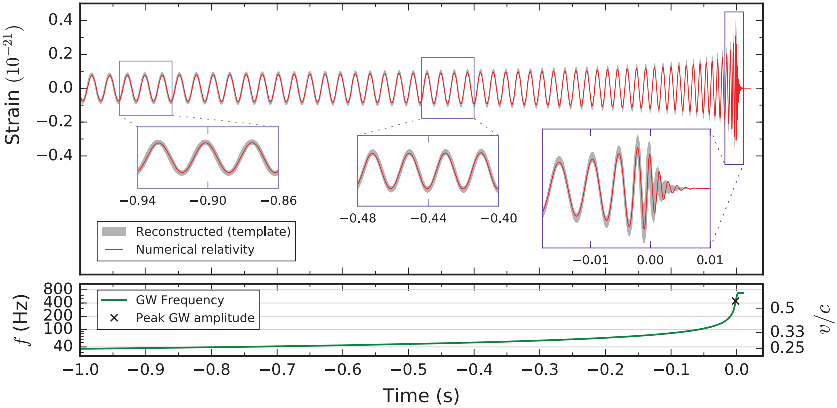
Lecture by M. A. Papa tomorrow morning

A recent example: the event GW151226



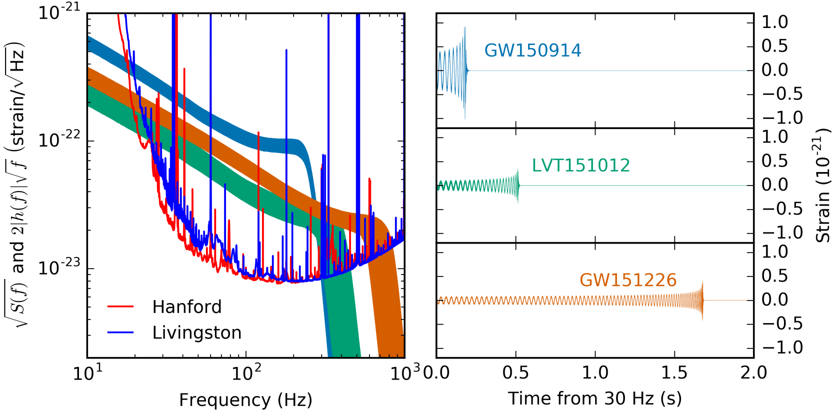
[PRL **116** (2016) 241103]

A long inspiral to merger to ringdown



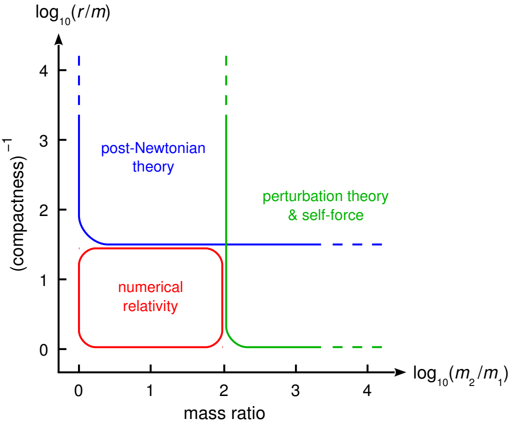
[PRL **116** (2016) 241103]

The first two/three detections



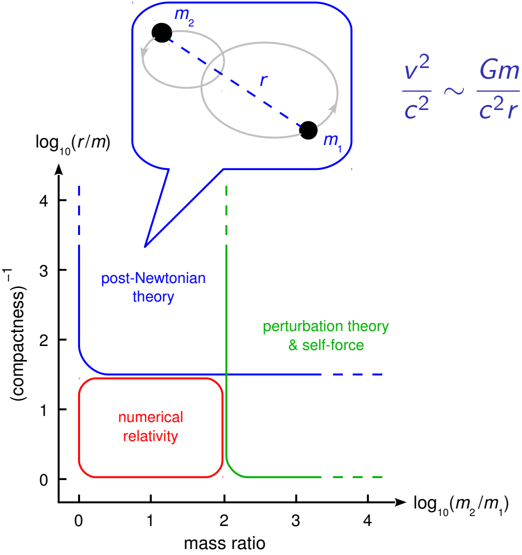
[gr-qc/1606.04856]

Modelling coalescing compact binaries



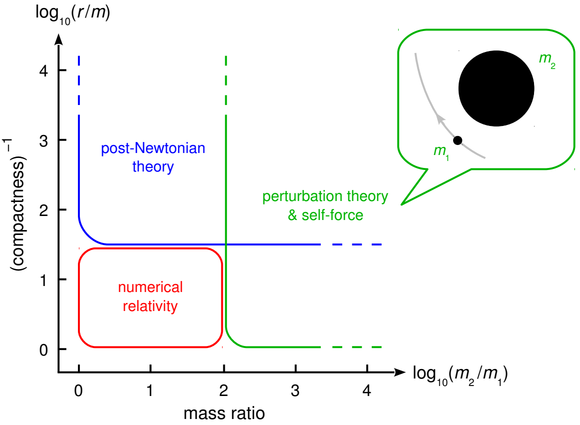
Modelling coalescing compact binaries

$$\frac{v^2}{c^2} \sim \frac{Gm}{c^2 r} \ll 1$$

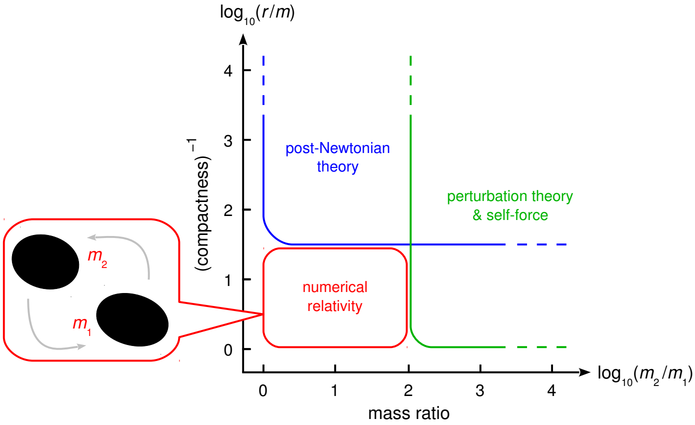


Modelling coalescing compact binaries

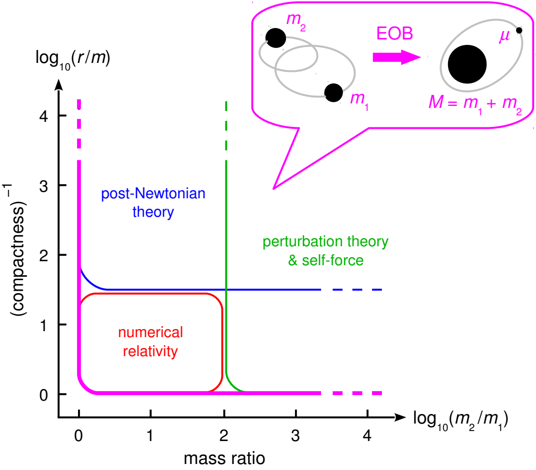
$$q \equiv \frac{m_1}{m_2} \ll 1$$



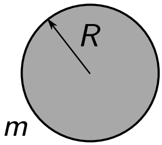
Modelling coalescing compact binaries



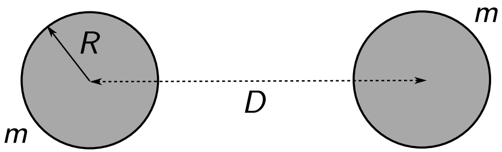
Modelling coalescing compact binaries



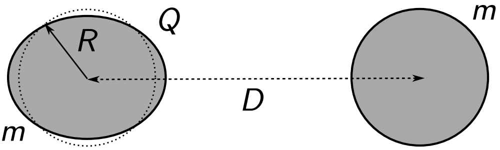
Effacement of the internal structure



Effacement of the internal structure

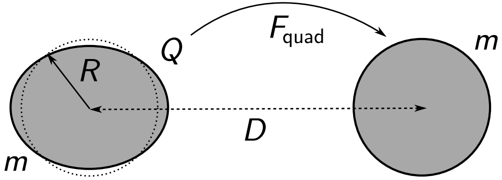


Effacement of the internal structure



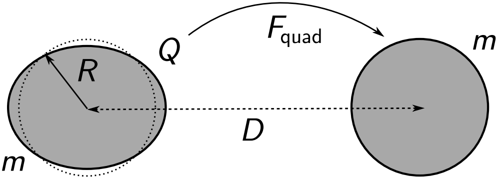
- Induced quadrupole moment: $Q_{ij} \sim R^5\, \partial_i \partial_j U \sim R^5 (Gm/D^3)$

Effacement of the internal structure



- Induced quadrupole moment: $Q_{ij} \sim R^5\, \partial_i \partial_j U \sim R^5 (Gm/D^3)$
- Induced quadrupolar force: $F^i_{\rm quad} \sim m\, \partial_i \left(\frac{Q}{r^3}\right) \sim R^5 (Gm^2/D^7)$

Effacement of the internal structure



- Induced quadrupole moment: $Q_{ij} \sim R^5\,\partial_i\partial_j U \sim R^5(Gm/D^3)$
- Induced quadrupolar force: $F^i_{\rm quad} \sim m\,\partial_i\big(\frac{Q}{r^3}\big) \sim R^5(Gm^2/D^7)$
- For a compact body with $R \sim Gm/c^2$,

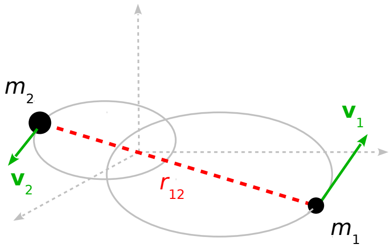
$$\frac{F_{\rm quad}}{F_{\rm Newt}} \sim \frac{(G^6/c^{10})(m/D)^7}{Gm^2/D^2} \sim \left(\frac{Gm}{c^2 D}\right)^5 \sim \left(\frac{v}{c}\right)^{10} \ll 1$$

Outline

1. Gravitational wave source modelling
2. Post-Newtonian approximation
3. Black hole perturbation theory
4. Effective one-body model
5. Comparisons

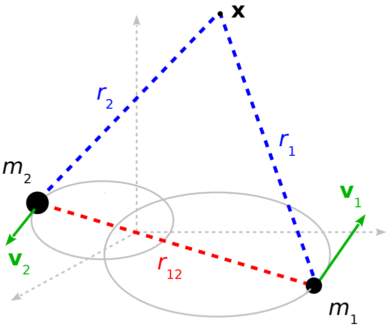
Small parameter

$$\varepsilon \sim \frac{\mathbf{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$



Small parameter

$$\varepsilon \sim \frac{\boldsymbol{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12} c^2} \ll 1$$

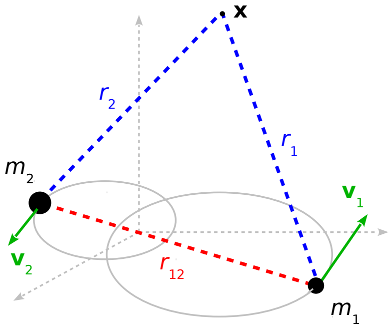


Example

$$g_{00}(t, \boldsymbol{x}) = -1 + \underbrace{\frac{2Gm_1}{r_1 c^2}}_{\text{Newtonian}} + \underbrace{\frac{4Gm_2 \boldsymbol{v}_2^2}{r_2 c^4}}_{\text{1PN term}} + \cdots + (1 \leftrightarrow 2)$$

Small parameter

$$\varepsilon \sim \frac{\boldsymbol{v}_{12}^2}{c^2} \sim \frac{Gm}{r_{12}c^2} \ll 1$$



Example

$$g_{00}(t,\boldsymbol{x}) = -1 + \underbrace{\frac{2Gm_1}{r_1 c^2}}_{\text{Newtonian}} + \underbrace{\frac{4Gm_2\boldsymbol{v}_2^2}{r_2 c^4}}_{\text{1PN term}} + \cdots + (1 \leftrightarrow 2)$$

Notation

nPN order refers to effects $\mathcal{O}(c^{-2n})$ with respect to "Newtonian" solution

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta}$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\,g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi\,|g|\,T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \end{cases}$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi \, |g| \, T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \equiv 16\pi \, \tau^{\alpha\beta} \end{cases}$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \;\Leftrightarrow\; \partial_\alpha \tau^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi \, |g| \, T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \equiv 16\pi \, \tau^{\alpha\beta} \end{cases}$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \;\Leftrightarrow\; \partial_\alpha \tau^{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha T^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi \,|g|\, T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \equiv 16\pi\, \tau^{\alpha\beta} \end{cases}$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \;\Leftrightarrow\; \partial_\alpha \tau^{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha T^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi\, |g|\, T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \equiv 16\pi\, \tau^{\alpha\beta} \end{cases}$$

Weak-field approximation

$$|h^{\alpha\beta}| \ll 1$$

Metric potential

$$h^{\alpha\beta} \equiv \sqrt{-g}\,g^{\alpha\beta} - \eta^{\alpha\beta}$$

Einstein field equations

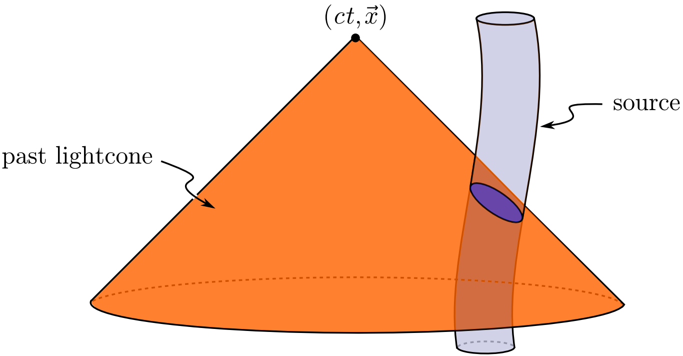
$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Longleftrightarrow \quad \begin{cases} \partial_\alpha h^{\alpha\beta} = 0 \;\Leftrightarrow\; \partial_\alpha \tau^{\alpha\beta} = 0 \;\Leftrightarrow\; \nabla_\alpha T^{\alpha\beta} = 0 \\ \Box h^{\alpha\beta} = 16\pi\,|g|\,T^{\alpha\beta} + \underbrace{\Lambda^{\alpha\beta}[h]}_{\substack{\text{nonlinearities} \\ \partial h \partial h + \cdots}} \equiv 16\pi\,\tau^{\alpha\beta} \end{cases}$$

Weak-field approximation

$$|h^{\alpha\beta}| \ll 1 \quad \Longrightarrow \quad \textbf{perturbative nonlinear treatment}$$

Flat space retarded propagator

$$h^{\alpha\beta}(t,\vec{x}) = -4\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\,\tau^{\alpha\beta}(t-|\vec{x}-\vec{y}|/c,\vec{y})$$



Post-Newtonian expansion

- For a post-Newtonian source of typical size d that evolves over a typical timescale T,

$$\frac{d}{\lambda_{\mathrm{GW}}} \sim \frac{vT}{c(T/2)} \sim \frac{v}{c} \ll 1$$

Post-Newtonian expansion

- For a post-Newtonian source of typical size d that evolves over a typical timescale T,

$$\frac{d}{\lambda_{\mathrm{GW}}} \sim \frac{vT}{c(T/2)} \sim \frac{v}{c} \ll 1$$

- Post-Newtonian expansion of an outgoing wave:

$$\frac{S(t-r/c)}{r} = \frac{S(t)}{r} - \frac{1}{c}\,\dot{S}(t) + \frac{r}{2c^2}\,\ddot{S}(t) - \frac{r^2}{6c^3}\,\dddot{S}(t) + \cdots$$

Post-Newtonian expansion

- For a post-Newtonian source of typical size d that evolves over a typical timescale T,

$$\frac{d}{\lambda_{\mathrm{GW}}} \sim \frac{vT}{c(T/2)} \sim \frac{v}{c} \ll 1$$

- Post-Newtonian expansion of an outgoing wave:

$$\frac{S(t-r/c)}{r} = \frac{S(t)}{r} - \underbrace{\frac{1}{c}\,\dot{S}(t)}_{\sim S/\lambda_{\mathrm{GW}}} + \underbrace{\frac{r}{2c^2}\,\ddot{S}(t)}_{\sim Sr/\lambda_{\mathrm{GW}}^2} - \underbrace{\frac{r^2}{6c^3}\,\dddot{S}(t)}_{\sim Sr^2/\lambda_{\mathrm{GW}}^3} + \cdots$$

Post-Newtonian expansion

- For a post-Newtonian source of typical size d that evolves over a typical timescale T,

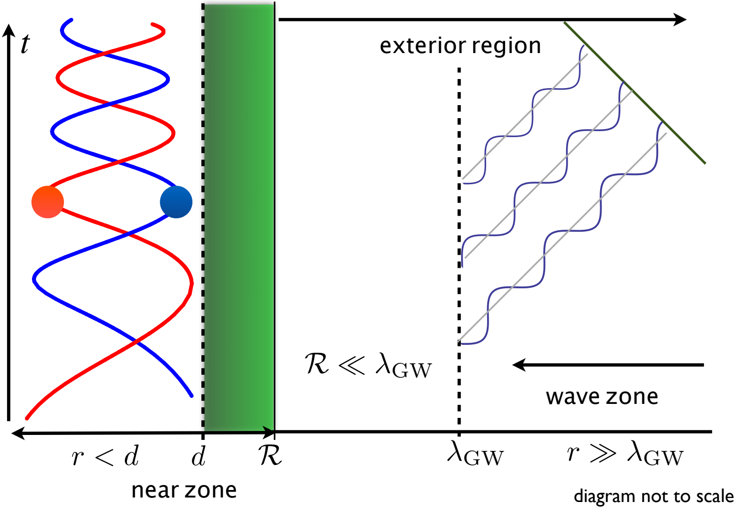
$$\frac{d}{\lambda_{\mathrm{GW}}} \sim \frac{vT}{c(T/2)} \sim \frac{v}{c} \ll 1$$

- Post-Newtonian expansion of an outgoing wave:

$$\frac{S(t-r/c)}{r} = \frac{S(t)}{r} - \underbrace{\frac{1}{c}\,\dot{S}(t)}_{\sim S/\lambda_{\mathrm{GW}}} + \underbrace{\frac{r}{2c^2}\,\ddot{S}(t)}_{\sim Sr/\lambda_{\mathrm{GW}}^2} - \underbrace{\frac{r^2}{6c^3}\,\dddot{S}(t)}_{\sim Sr^2/\lambda_{\mathrm{GW}}^3} + \cdots$$

Expansion ill-behaved when $r \gtrsim \lambda_{\mathrm{GW}}$

A wave generation formalism

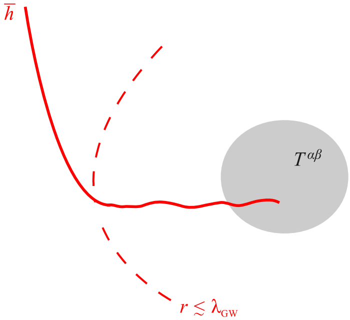


(Credit: Buonanno & Sathyaprakash 2015)

A wave generation formalism

- Post-Newtonian expansion in *near-zone* region $r \ll \lambda_{\mathrm{GW}}$:

$$h_{\mathrm{PN}} \equiv \overline{h} = \sum_{n\geqslant 0} c^{-n} h^{\mathrm{PN}}_{(n)}$$



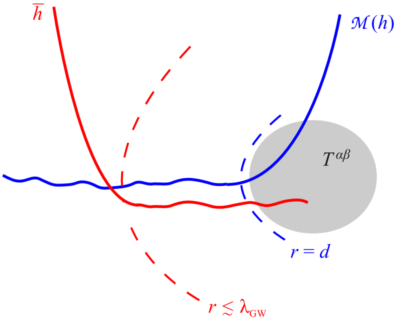
A wave generation formalism

- Post-Newtonian expansion in *near-zone* region $r \ll \lambda_{\mathrm{GW}}$:

$$h_{\mathrm{PN}} \equiv \overline{h} = \sum_{n \geqslant 0} c^{-n} h^{\mathrm{PN}}_{(n)}$$

- Post-Minkowskian expansion in *exterior* region $r > d$:

$$h_{\mathrm{PM}} \equiv \mathcal{M}(h) = \sum_{k \geqslant 1} G^k h^{\mathrm{PM}}_{(k)}$$



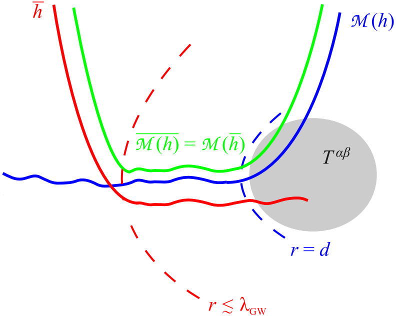
A wave generation formalism

- Post-Newtonian expansion in *near-zone* region $r \ll \lambda_{\rm GW}$:

$$h_{\rm PN} \equiv \overline{h} = \sum_{n\geqslant 0} c^{-n} h^{\rm PN}_{(n)}$$

- Post-Minkowskian expansion in *exterior* region $r > d$:

$$h_{\rm PM} \equiv \mathcal{M}(h) = \sum_{k\geqslant 1} G^k h^{\rm PM}_{(k)}$$



- Matching of asymptotic expansions in overlap region $d < r \ll \lambda_{\rm GW}$:

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h})$$

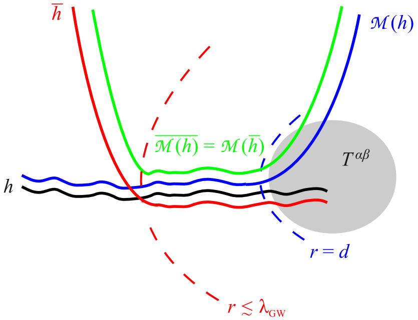
A wave generation formalism

- Post-Newtonian expansion in *near-zone* region $r \ll \lambda_{\mathrm{GW}}$:

$$h_{\mathrm{PN}} \equiv \overline{h} = \sum_{n \geqslant 0} c^{-n} h^{\mathrm{PN}}_{(n)}$$

- Post-Minkowskian expansion in *exterior* region $r > d$:

$$h_{\mathrm{PM}} \equiv \mathcal{M}(h) = \sum_{k \geqslant 1} G^k h^{\mathrm{PM}}_{(k)}$$



- Matching of asymptotic expansions in overlap region $d < r \ll \lambda_{\mathrm{GW}}$:

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h})$$

Extended bodies and point particles

- Gravitational field *not weak* in and near compact objects!

Extended bodies and point particles

- Gravitational field *not weak* in and near compact objects!
- **Strong equivalence principle** $\rightarrow$ PN approximation can be applied to strongly gravitating bodies (checked explicitly up to 2PN order)

Extended bodies and point particles

- Gravitational field *not weak* in and near compact objects!
- **Strong equivalence principle** $\rightarrow$ PN approximation can be applied to strongly gravitating bodies (checked explicitly up to 2PN order)
- Model such extended compact bodies as massive **point particles**

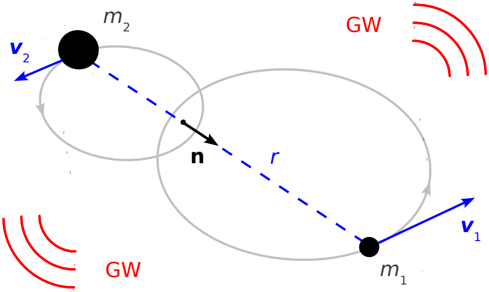
Extended bodies and point particles

- Gravitational field *not weak* in and near compact objects!
- **Strong equivalence principle** $\rightarrow$ PN approximation can be applied to strongly gravitating bodies (checked explicitly up to 2PN order)
- Model such extended compact bodies as massive **point particles**
- Divergent self-field of a point particle $\rightarrow$ *regularization* scheme

Extended bodies and point particles

- Gravitational field *not weak* in and near compact objects!
- **Strong equivalence principle** $\rightarrow$ PN approximation can be applied to strongly gravitating bodies (checked explicitly up to 2PN order)
- Model such extended compact bodies as massive **point particles**
- Divergent self-field of a point particle $\rightarrow$ *regularization* scheme
- Each point mass moves along a geodesic of a *regularized* metric

Post-Newtonian equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \underbrace{-\frac{G m_2}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4}}_{\text{conservative terms}} + \underbrace{\frac{\boldsymbol{A}_{2.5\mathrm{PN}}}{c^5}}_{\text{rad. reac.}} + \underbrace{\frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6}}_{\text{cons. term}} + \underbrace{\frac{\boldsymbol{A}_{3.5\mathrm{PN}}}{c^7}}_{\text{rad. reac.}} + \cdots$$

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n}$$

0PN	Newton (1687)

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{\text{1PN}}}{c^2}$$

0PN	Newton (1687)
1PN	Lorentz & Droste (1917); Einstein *et al.* (1938)

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4}$$

0PN	Newton (1687)
1PN	Lorentz & Droste (1917); Einstein *et al.* (1938)
2PN	Damour & Deruelle (1982)

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4} + \frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6}$$

0PN	Newton (1687)
1PN	Lorentz & Droste (1917); Einstein *et al.* (1938)
2PN	Damour & Deruelle (1982)
3PN	Jaranowski & Schäfer (1999); Blanchet & Faye (2001) Itoh & Futamase (2003); Foffa & Sturani (2011)

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4} + \frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6} + \underbrace{\frac{\boldsymbol{A}_{4\mathrm{PN}}}{c^8}}_{\text{non local}} + \cdots$$

0PN	Newton (1687)
1PN	Lorentz & Droste (1917); Einstein *et al.* (1938)
2PN	Damour & Deruelle (1982)
3PN	Jaranowski & Schäfer (1999); Blanchet & Faye (2001) Itoh & Futamase (2003); Foffa & Sturani (2011)
4PN	Jaranowski & Schäfer (2013); Damour *et al.* (2014) Bernard, Blanchet *et al.* (2016)

Conservative post-Newtonian dynamics

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\,\boldsymbol{n} + \frac{\boldsymbol{A}_{1\mathrm{PN}}}{c^2} + \frac{\boldsymbol{A}_{2\mathrm{PN}}}{c^4} + \frac{\boldsymbol{A}_{3\mathrm{PN}}}{c^6} + \underbrace{\frac{\boldsymbol{A}_{4\mathrm{PN}}}{c^8}}_{\text{non local}} + \cdots$$

0PN	Newton (1687)
1PN	Lorentz & Droste (1917); Einstein *et al.* (1938)
2PN	Damour & Deruelle (1982)
3PN	Jaranowski & Schäfer (1999); Blanchet & Faye (2001) Itoh & Futamase (2003); Foffa & Sturani (2011)
4PN	Jaranowski & Schäfer (2013); Damour *et al.* (2014) Bernard, Blanchet *et al.* (2016)

Poincaré group symmetries $\longrightarrow$ 10 conserved quantities

Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\to$ 4PN binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}(m\omega)^{2/3}}_{\text{Newtonian binding energy}} \underbrace{\Big(1 + \cdots\Big)}_{\text{4PN relative correction}}$$

Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\rightarrow$ 4PN binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}(m\omega)^{2/3}}_{\substack{\text{Newtonian}\\ \text{binding energy}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{4PN relative}\\ \text{correction}}}$$

- Wave generation formalism $\rightarrow$ 3.5PN GW energy flux

$$\mathcal{F}(\omega) = \underbrace{\frac{32}{5}\frac{\mu^2}{m^2}(m\omega)^5}_{\substack{\text{Einstein's}\\ \text{quad. formula}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{3.5PN relative}\\ \text{correction}}}$$

Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\rightarrow$ 4PN binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}(m\omega)^{2/3}}_{\substack{\text{Newtonian}\\ \text{binding energy}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{4PN relative}\\ \text{correction}}}$$

- Wave generation formalism $\rightarrow$ 3.5PN GW energy flux

$$\mathcal{F}(\omega) = \underbrace{\frac{32}{5}\frac{\mu^2}{m^2}(m\omega)^5}_{\substack{\text{Einstein's}\\ \text{quad. formula}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{3.5PN relative}\\ \text{correction}}}$$

- Energy balance $\rightarrow$ 3.5PN orbital phase and GW phase

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$$

Phasing for inspiralling compact binaries

- Conservative orbital dynamics $\rightarrow$ 4PN binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}\,(m\omega)^{2/3}}_{\substack{\text{Newtonian}\\ \text{binding energy}}}\;\underbrace{\Big(1+\cdots\Big)}_{\substack{\text{4PN relative}\\ \text{correction}}}$$

- Wave generation formalism $\rightarrow$ 3.5PN GW energy flux

$$\mathcal{F}(\omega) = \underbrace{\frac{32}{5}\,\frac{\mu^2}{m^2}\,(m\omega)^5}_{\substack{\text{Einstein's}\\ \text{quad. formula}}}\;\underbrace{\Big(1+\cdots\Big)}_{\substack{\text{3.5PN relative}\\ \text{correction}}}$$

- Energy balance $\rightarrow$ 3.5PN orbital phase and GW phase

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \implies \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\mathcal{F}(\omega)}{E'(\omega)}$$

Phasing for inspiralling compact binaries

- Conservative orbital dynamics → 4PN binding energy

$$E(\omega) = \underbrace{-\frac{\mu}{2}(m\omega)^{2/3}}_{\substack{\text{Newtonian}\\ \text{binding energy}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{4PN relative}\\ \text{correction}}}$$

- Wave generation formalism → 3.5PN GW energy flux

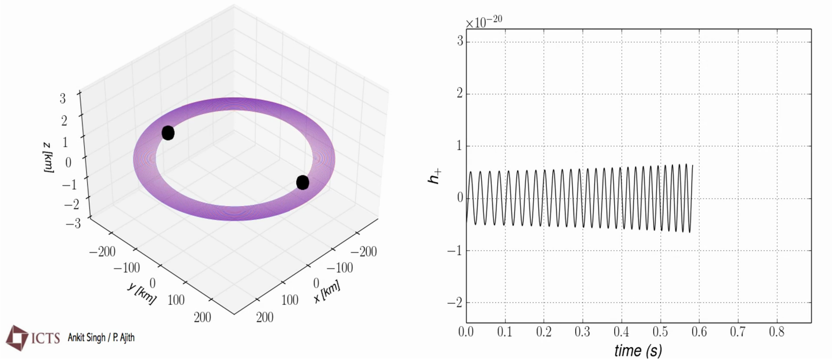
$$\mathcal{F}(\omega) = \underbrace{\frac{32}{5}\frac{\mu^2}{m^2}(m\omega)^5}_{\substack{\text{Einstein's}\\ \text{quad. formula}}} \underbrace{\Big(1 + \cdots\Big)}_{\substack{\text{3.5PN relative}\\ \text{correction}}}$$

- Energy balance → 3.5PN orbital phase and GW phase

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} \implies \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\mathcal{F}(\omega)}{E'(\omega)} \implies \phi(t) = \int^t \omega(t')\,\mathrm{d}t'$$

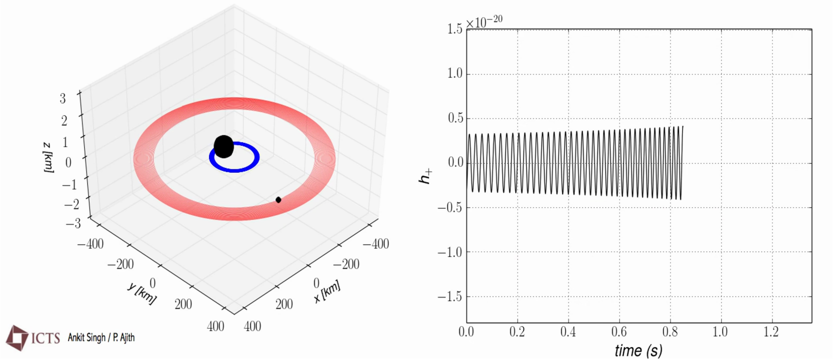
Waveform for inspiralling compact binaries

Equal masses

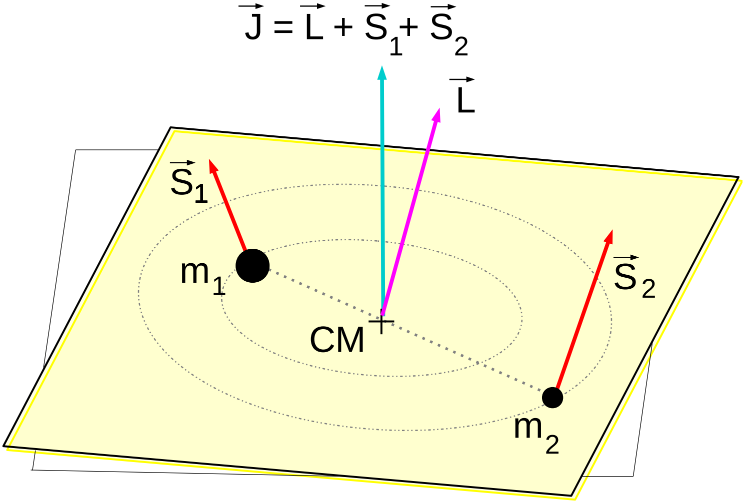


Waveform for inspiralling compact binaries

Unequal masses



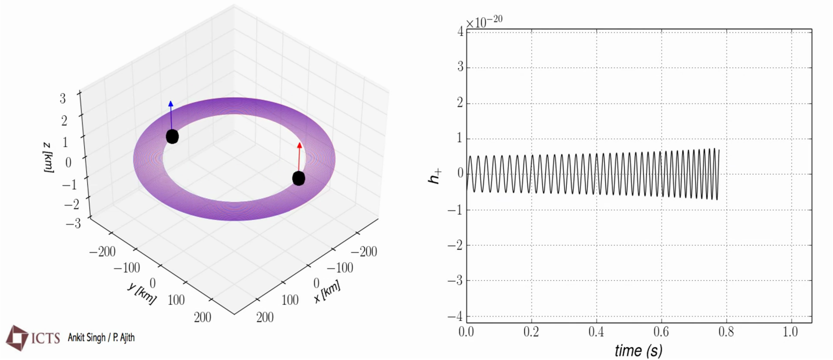
Binary systems of spinning compact bodies



(Credit: L. Blanchet)

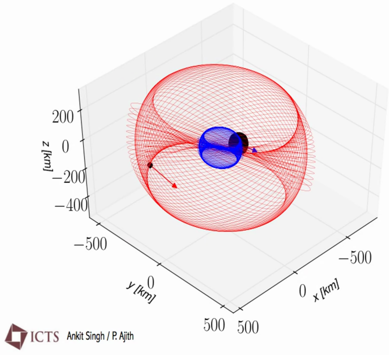
Spin effects on the waveform

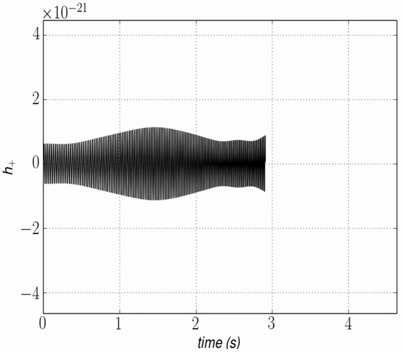
Equal masses and aligned spins



Spin effects on the waveform

Unequal masses and misaligned spins



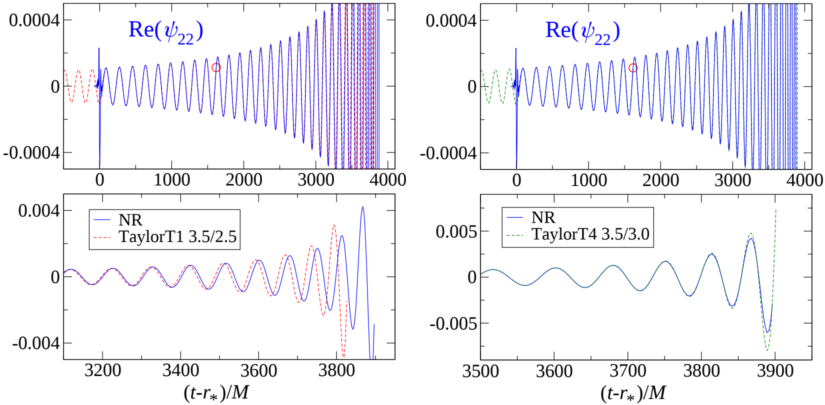


State of the art

	Spinless	Spin-Orbit	Spin-Squared	Tidal
Conserv. dynamics	4PN	3.5PN	3PN	7PN
Energy flux	3.5PN	4PN	2PN	6PN
Radiation reaction	4.5PN	4PN	4.5PN	6PN
Waveform phase	3.5PN	4PN	2PN	6PN
Waveform amplitude	3PN	2PN	2PN	6PN

PN *vs* NR waveforms

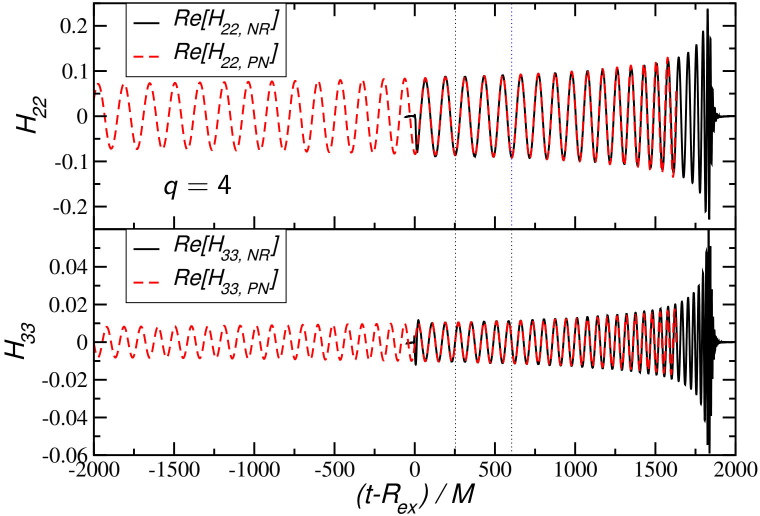
Equal masses and no spins



[PRD **76** (2007) 124038]

PN *vs* NR waveforms

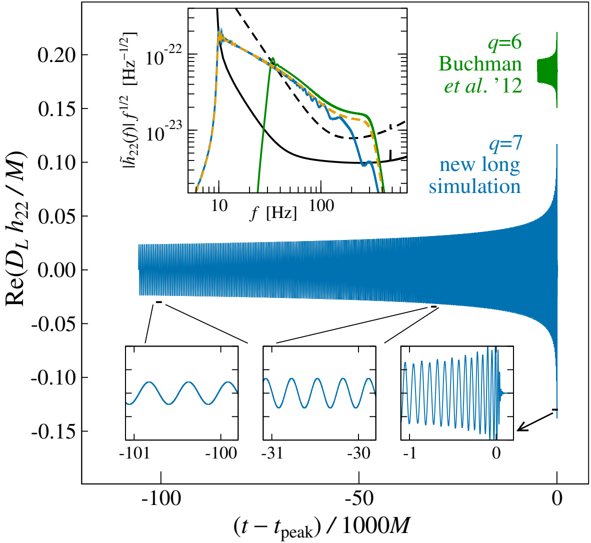
Unequal masses and no spins



[CQG **28** (2011) 134004]

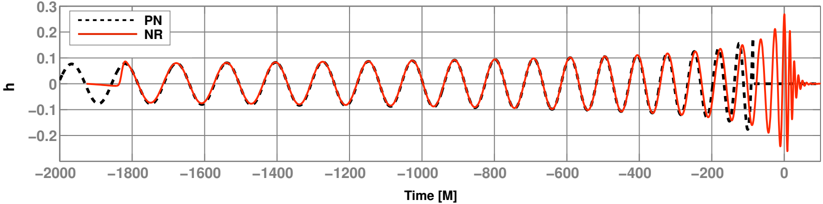
350 GW cycles!

PN *vs* NR waveforms



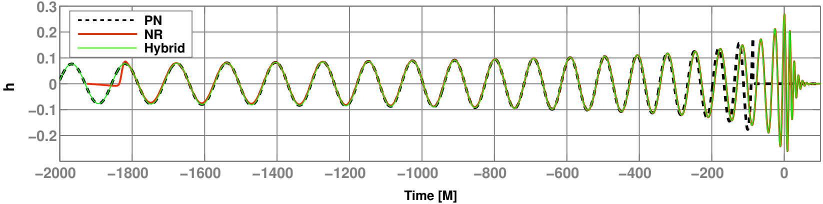
[PRL **115** (2015) 031102]

Hybrid PN/NR waveforms



[PRD **77** (2008) 104017]

Hybrid PN/NR waveforms



[PRD **77** (2008) 104017]

Further reading

Review articles

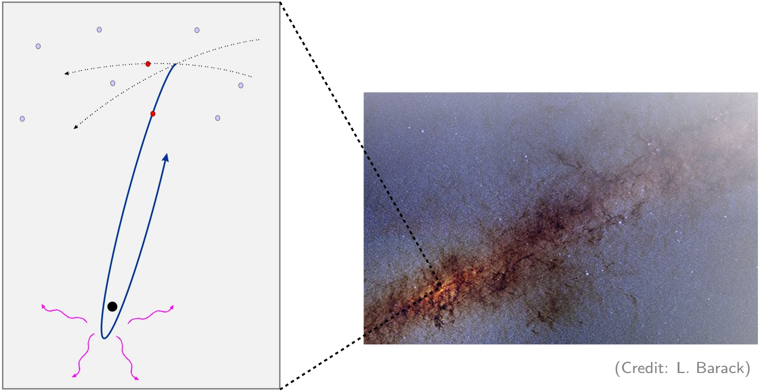
- *Gravitational radiation from post-Newtonian sources...*
  L. Blanchet, Living Rev. Rel. **17**, 2 (2014)
- *Post-Newtonian methods: Analytic results on the binary problem*
  G. Schäfer, in *Mass and motion in general relativity*
  Edited by L. Blanchet et al., Springer (2011)
- *The post-Newtonian approximation for relativistic compact binaries*
  T. Futamase and Y. Itoh, Living Rev. Rel. **10**, 2 (2007)

Topical books

- *Gravity: Newtonian, post-Newtonian, relativistic*
  E. Poisson and C. M. Will, Cambridge University Press (2015)
- *Gravitational waves: Theory and experiments*
  M. Maggiore, Oxford University Press (2007)

Outline

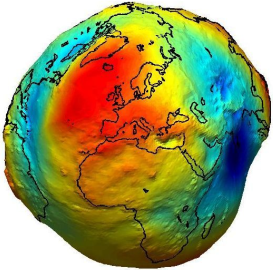
Extreme mass ratio inspirals (EMRIs)



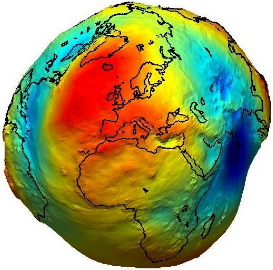
(Credit: L. Barack)

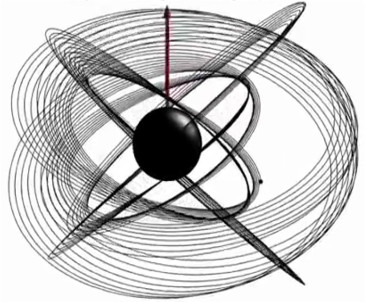
- eLISA sensitive to $M_{\rm BH} \sim 10^5 - 10^7 M_\odot \rightarrow q \sim 10^{-7} - 10^{-4}$
- $T_{\rm orb} \propto M_{\rm BH} \sim$ hr and $T_{\rm insp} \propto M_{\rm BH}/q \sim$ yrs

Geodesy



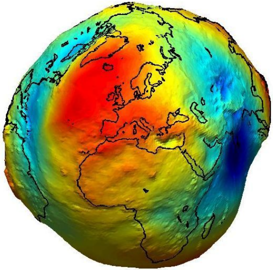
Geodesy



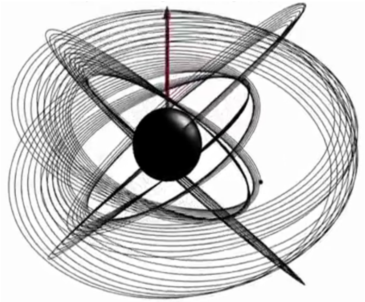


(Credit: S. Drasco)

Geodesy

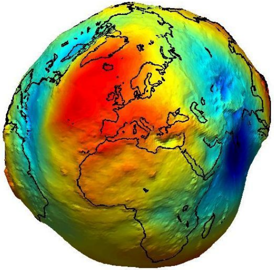


Botriomeladesy

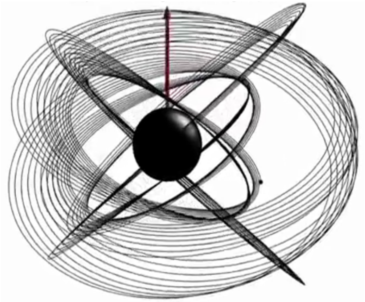


(Credit: S. Drasco)

Geodesy



Botriomeladesy

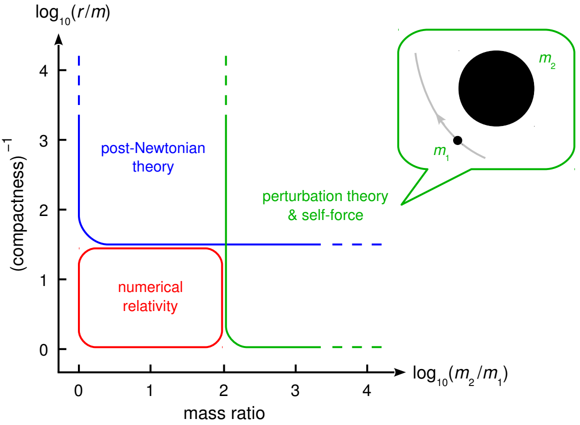


(Credit: S. Drasco)

Test of the black hole no hair theorem

Large to extreme mass ratios

$$q \equiv \frac{m_1}{m_2} \ll 1$$



Black hole perturbation theory

Metric perturbation

$$h_{\alpha\beta} \equiv \underbrace{\mathfrak{g}_{\alpha\beta}}_{\text{exact}} - \underbrace{g_{\alpha\beta}}_{\text{bkgd}} = \mathcal{O}(q)$$

Black hole perturbation theory

Metric perturbation

$$h_{\alpha\beta} \equiv \underbrace{\mathfrak{g}_{\alpha\beta}}_{\text{exact}} - \underbrace{g_{\alpha\beta}}_{\text{bkgd}} = \mathcal{O}(q)$$

Lorenz gauge condition

$$\nabla^\alpha \bar{h}_{\alpha\beta} = 0$$

Black hole perturbation theory

Metric perturbation

$$h_{\alpha\beta} \equiv \underbrace{\mathfrak{g}_{\alpha\beta}}_{\text{exact}} - \underbrace{g_{\alpha\beta}}_{\text{bkgd}} = \mathcal{O}(q)$$

Lorenz gauge condition

$$\nabla^{\alpha}\bar{h}_{\alpha\beta} = 0$$

Einstein field equations

$$\Box_g \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

Black hole perturbation theory

Metric perturbation

$$h_{\alpha\beta} \equiv \underbrace{\mathfrak{g}_{\alpha\beta}}_{\text{exact}} - \underbrace{g_{\alpha\beta}}_{\text{bkgd}} = \mathcal{O}(q)$$

Lorenz gauge condition

$$\nabla^{\alpha}\bar{h}_{\alpha\beta} = 0$$

Einstein field equations

$$\Box_g \bar{h}_{\alpha\beta} + 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\bar{h}_{\mu\nu} = -16\pi T_{\alpha\beta}$$

Linear equation but involved Green's function

Black hole perturbation theory

Schwarzschild

- Spherical symmetry $\rightarrow$ spherical harmonics $Y_{\ell m}(\theta, \phi)$
- Staticity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Regge-Wheeler-Zerilli-Moncrief formalism

Black hole perturbation theory

Schwarzschild

- Spherical symmetry $\rightarrow$ spherical harmonics $Y_{\ell m}(\theta, \phi)$
- Staticity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Regge-Wheeler-Zerilli-Moncrief formalism

Kerr

- Stationarity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Axial symmetry $\rightarrow$ spin-weighted spheroidal harm. ${}_{-2}S^{a\omega}_{\ell m}(\theta)e^{\mathrm{i}m\phi}$
- Teukolsky equation for ψ_0/ψ_4 is **separable**

Black hole perturbation theory

Schwarzschild

- Spherical symmetry $\rightarrow$ spherical harmonics $Y_{\ell m}(\theta, \phi)$
- Staticity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Regge-Wheeler-Zerilli-Moncrief formalism

Kerr

- Stationarity $\rightarrow$ Fourier mode decomposition $e^{-\mathrm{i}\omega t}$
- Axial symmetry $\rightarrow$ spin-weighted spheroidal harm. ${}_{-2}S^{a\omega}_{\ell m}(\theta) e^{\mathrm{i}m\phi}$
- Teukolsky equation for ψ_0/ψ_4 is **separable**

Gravitational waveform + Fluxes

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Constants of the motion

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Constants of the motion

- Energy $E = -t^\alpha u_\alpha$

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x,u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Constants of the motion

- Energy $E = -t^\alpha u_\alpha$
- Ang. momentum $L_z = \phi^\alpha u_\alpha$

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Constants of the motion

- Energy $E = -t^\alpha u_\alpha$
- Ang. momentum $L_z = \phi^\alpha u_\alpha$
- Carter constant $Q = K^{\alpha\beta} u_\alpha u_\beta$

Geodesic motion in Kerr spacetime

Canonical Hamiltonian

$$H(x, u) = \frac{1}{2} g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Completely integrable

Constants of the motion

- Energy $E = -t^\alpha u_\alpha$
- Ang. momentum $L_z = \phi^\alpha u_\alpha$
- Carter constant $Q = K^{\alpha\beta} u_\alpha u_\beta$

Geodesic motion in Kerr spacetime

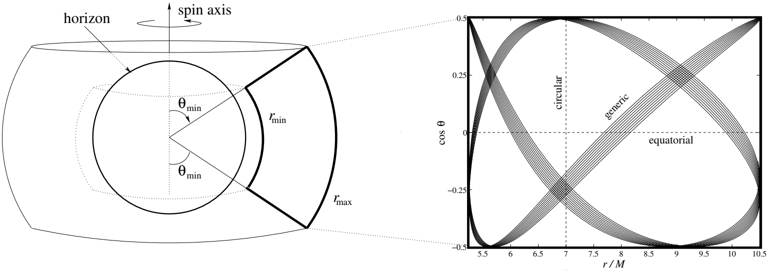
Canonical Hamiltonian

$$H(x,u) = \frac{1}{2}\, g^{\alpha\beta}(x)\, u_\alpha u_\beta$$

Completely integrable

Constants of the motion

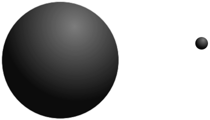
- Energy $E = -t^\alpha u_\alpha$
- Ang. momentum $L_z = \phi^\alpha u_\alpha$
- Carter constant $Q = K^{\alpha\beta} u_\alpha u_\beta$



(Credit: Drasco & Hughes 2006)

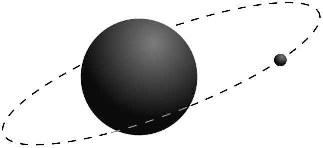
Adiabatic approximation

$$\frac{T_{\text{insp}}}{T_{\text{orb}}} \propto \frac{1}{q} \gg 1$$



Adiabatic approximation

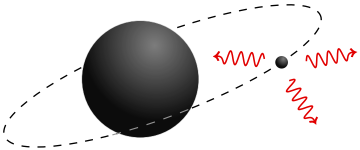
$$\frac{T_{\text{insp}}}{T_{\text{orb}}} \propto \frac{1}{q} \gg 1$$



- Choose a geodesic orbit (E, L_z, Q) for the point-mass source

Adiabatic approximation

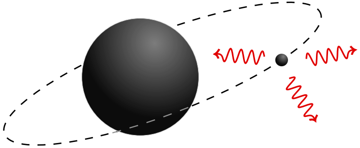
$$\frac{T_{\text{insp}}}{T_{\text{orb}}} \propto \frac{1}{q} \gg 1$$



- Choose a geodesic orbit (E, L_z, Q) for the point-mass source
- Compute the resulting gravitational waves $h_{+,\times}$ and fluxes $\mathcal{F}$

Adiabatic approximation

$$\frac{T_{\mathrm{insp}}}{T_{\mathrm{orb}}} \propto \frac{1}{q} \gg 1$$

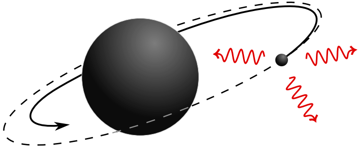


- Choose a geodesic orbit (E, L_z, Q) for the point-mass source
- Compute the resulting gravitational waves $h_{+,\times}$ and fluxes $\mathcal{F}$
- Impose **balance** of energy and angular momentum:

$$\langle \dot{E} \rangle = -\mathcal{F}_E \,, \quad \langle \dot{L}_z \rangle = -\mathcal{F}_{L_z} \,, \quad \langle \dot{Q} \rangle = ?$$

Adiabatic approximation

$$\frac{T_{\rm insp}}{T_{\rm orb}} \propto \frac{1}{q} \gg 1$$

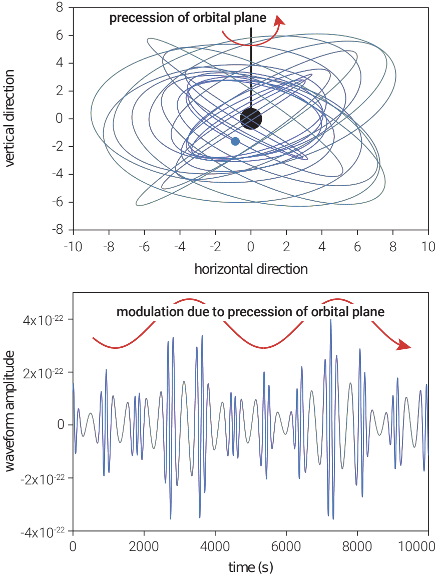


- Choose a geodesic orbit (E, L_z, Q) for the point-mass source
- Compute the resulting gravitational waves $h_{+,\times}$ and fluxes $\mathcal{F}$
- Impose **balance** of energy and angular momentum:

$$\langle \dot{E} \rangle = -\mathcal{F}_E\,, \quad \langle \dot{L}_z \rangle = -\mathcal{F}_{L_z}\,, \quad \langle \dot{Q} \rangle = ?$$

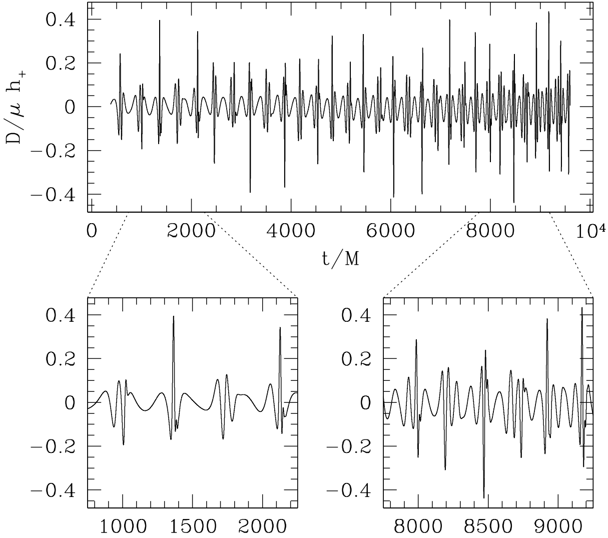
- **Update** the orbit and play again!

Waveform in the adiabatic approximation



[eLISA whitepaper]

Waveform in the adiabatic approximation



[PRD **78** (2008) 024022]

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\rm insp} \sim M_{\rm BH}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q}\left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2)\right]$$

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\rm insp} \sim M_{\rm BH}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q}\left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2)\right]$$

- Using ϕ_0 is likely good enough for signal detection

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\mathrm{insp}} \sim M_{\mathrm{BH}}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q}\left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2)\right]$$

- Using ϕ_0 is likely good enough for signal detection
- Including ϕ_1 will be enough for parameter estimation

Accuracy requirement for the phasing

- Over an inspiral timescale $T_{\mathrm{insp}} \sim M_{\mathrm{BH}}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q} \left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2) \right]$$

- Using ϕ_0 is likely good enough for signal detection
- Including ϕ_1 will be enough for parameter estimation
- But the adiabatic approximation only gives access to ϕ_0

Accuracy requirement for the phasing

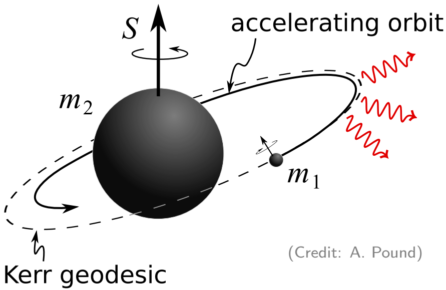
- Over an inspiral timescale $T_{\rm insp} \sim M_{\rm BH}/q$, the GW phase is given by the expansion

$$\phi = \frac{1}{q}\left[\phi_0 + q\,\phi_1 + \mathcal{O}(q^2)\right]$$

- Using ϕ_0 is likely good enough for signal detection
- Including ϕ_1 will be enough for parameter estimation
- But the adiabatic approximation only gives access to ϕ_0

We need to account for the local effects of the metric perturbation on the body's orbital motion

Gravitational self-force



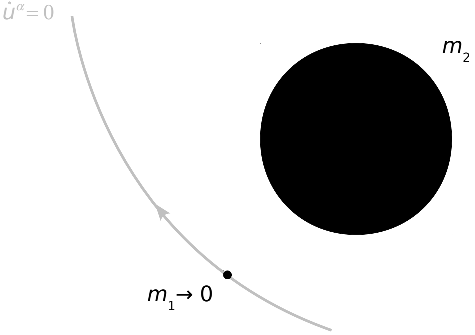
(Credit: A. Pound)

- Dissipative component $\longleftrightarrow$ gravitational waves
- Conservative component $\longleftrightarrow$ secular effects

Gravitational self-force

Spacetime metric

$$\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta}$$



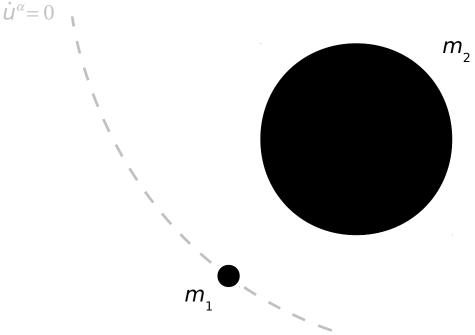
Gravitational self-force

Spacetime metric

$$\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta}$$

Small parameter

$$q \equiv \frac{m_1}{m_2} \ll 1$$



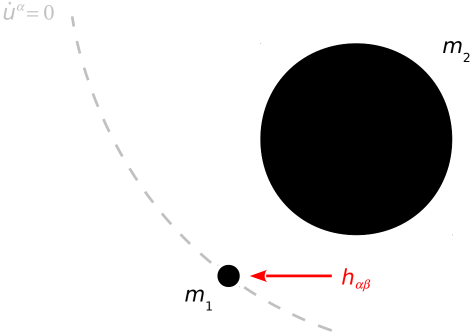
Gravitational self-force

Spacetime metric

$$\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

Small parameter

$$q \equiv \frac{m_1}{m_2} \ll 1$$



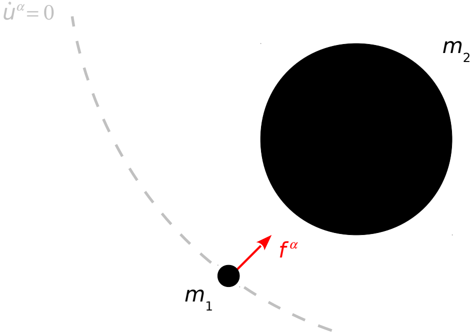
Gravitational self-force

Spacetime metric

$$\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

Small parameter

$$q \equiv \frac{m_1}{m_2} \ll 1$$



Gravitational self-force

Spacetime metric

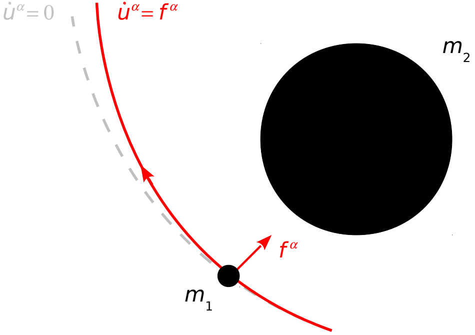
$$\mathfrak{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

Small parameter

$$q \equiv \frac{m_1}{m_2} \ll 1$$

Gravitational self-force

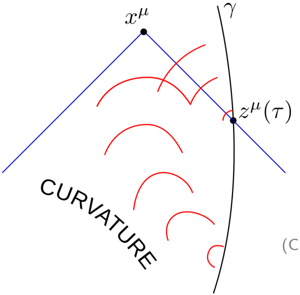
$$\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha = f^\alpha$$



Equation of motion

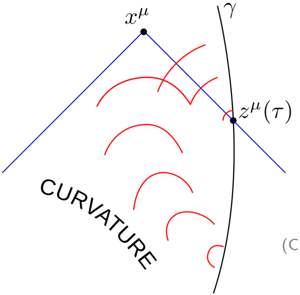
Metric perturbation

$$h_{\alpha\beta} = h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$



(Credit: A. Pound)

Equation of motion



(Credit: A. Pound)

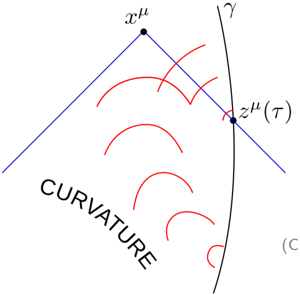
Metric perturbation

$$h_{\alpha\beta} = h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$

MiSaTaQuWa equation

$$\dot{u}^\alpha = -\underbrace{\left(g^{\alpha\beta} + u^\alpha u^\beta\right)}_{\text{projector} \perp u^\alpha}\underbrace{\left(\nabla_\lambda h^{\text{tail}}_{\beta\sigma} - \frac{1}{2}\nabla_\beta h^{\text{tail}}_{\lambda\sigma}\right)}_{\text{“force”}} u^\lambda u^\sigma$$

Equation of motion



(Credit: A. Pound)

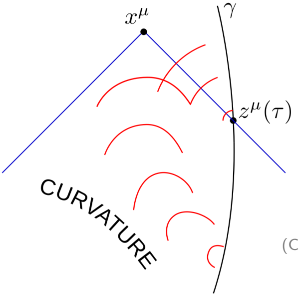
Metric perturbation

$$h_{\alpha\beta} = h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$

MiSaTaQuWa equation

$$\dot{u}^\alpha = -\underbrace{\left(g^{\alpha\beta} + u^\alpha u^\beta\right)}_{\text{projector } \perp\, u^\alpha} \underbrace{\left(\nabla_\lambda h^{\text{tail}}_{\beta\sigma} - \frac{1}{2}\nabla_\beta h^{\text{tail}}_{\lambda\sigma}\right)}_{\text{"force"}} u^\lambda u^\sigma \equiv f^\alpha[h^{\text{tail}}]$$

Equation of motion



(Credit: A. Pound)

Metric perturbation

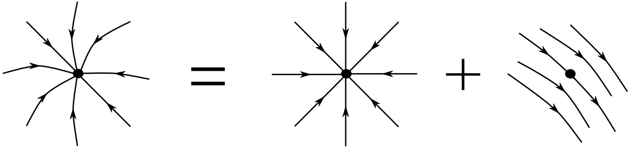
$$h_{\alpha\beta} = h^{\text{direct}}_{\alpha\beta} + h^{\text{tail}}_{\alpha\beta}$$

MiSaTaQuWa equation

$$\dot{u}^\alpha = -\underbrace{\left(g^{\alpha\beta} + u^\alpha u^\beta\right)}_{\text{projector } \perp u^\alpha}\underbrace{\left(\nabla_\lambda h^{\text{tail}}_{\beta\sigma} - \frac{1}{2}\nabla_\beta h^{\text{tail}}_{\lambda\sigma}\right)}_{\text{“force”}} u^\lambda u^\sigma \equiv f^\alpha[h^{\text{tail}}]$$

Beware: the self-force is *gauge-dependant*

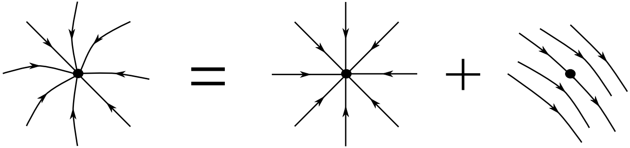
Generalized equivalence principle



(Credit: A. Pound)

body's field $h_{\alpha\beta}$ = singular field $h^S_{\alpha\beta}$ + regular field $h^R_{\alpha\beta}$

Generalized equivalence principle



(Credit: A. Pound)

body's field $h_{\alpha\beta}$ = singular field $h^S_{\alpha\beta}$ + regular field $h^R_{\alpha\beta}$

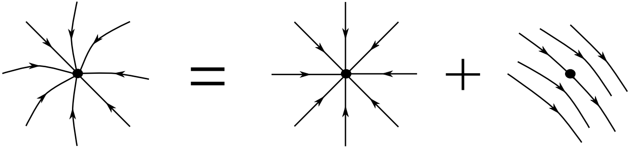
singular/self field

$$h^S \sim m/r$$

$$\Box h^S \sim -16\pi T$$

$$f^\alpha[h^S] = 0$$

Generalized equivalence principle



(Credit: A. Pound)

body's field $h_{\alpha\beta}$ = singular field $h^S_{\alpha\beta}$ + regular field $h^R_{\alpha\beta}$

singular/self field

$$h^S \sim m/r$$

$$\Box h^S \sim -16\pi T$$

$$f^\alpha[h^S] = 0$$

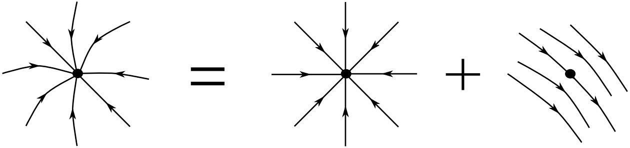
regular/residual field

$$h^R \sim h^{\text{tail}} + \text{local terms}$$

$$\Box h^R \sim 0$$

$$\dot{u}^\alpha = f^\alpha[h^R]$$

Generalized equivalence principle



(Credit: A. Pound)

body's field $h_{\alpha\beta}$ singular field $h^S_{\alpha\beta}$ regular field $h^R_{\alpha\beta}$

singular/self field

$$h^S \sim m/r$$

$$\Box h^S \sim -16\pi T$$

$$f^\alpha[h^S] = 0$$

regular/residual field

$$h^R \sim h^{\text{tail}} + \text{local terms}$$

$$\Box h^R \sim 0$$

$$\dot{u}^\alpha = f^\alpha[h^R]$$

Self-acc. motion in $g_{\alpha\beta}$ $\iff$ **Geodesic motion** in $g_{\alpha\beta} + h^R_{\alpha\beta}$

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]
- Practical calculations for generic orbits in **Schwarzschild**
  [Barack & Sago (2010); Warburton et al. (2012); Osburn et al. (2016)]

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]
- Practical calculations for generic orbits in **Schwarzschild**
  [Barack & Sago (2010); Warburton et al. (2012); Osburn et al. (2016)]
- **Gauge-invariant** effects of the conservative self-force
  [Barack & Sago (2009); Shah et al. (2012); Isoyama et al. (2014); . . .]

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]
- Practical calculations for generic orbits in **Schwarzschild**
  [Barack & Sago (2010); Warburton et al. (2012); Osburn et al. (2016)]
- **Gauge-invariant** effects of the conservative self-force
  [Barack & Sago (2009); Shah et al. (2012); Isoyama et al. (2014); ...]
- Practical calculations for equatorial orbits in **Kerr**
  [van de Meent & Shah (2015); van de Meent (2016)]

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]
- Practical calculations for generic orbits in **Schwarzschild**
  [Barack & Sago (2010); Warburton et al. (2012); Osburn et al. (2016)]
- **Gauge-invariant** effects of the conservative self-force
  [Barack & Sago (2009); Shah et al. (2012); Isoyama et al. (2014); . . .]
- Practical calculations for equatorial orbits in **Kerr**
  [van de Meent & Shah (2015); van de Meent (2016)]
- Formulation of **second-order** gravitational self-force
  [Detweiler (2012); Gralla (2012); Pound (2012-2015)]

Recent developments

- **Rigorous** formulation of gravitational self-force
  [Gralla & Wald (2008); Pound (2010); Harte (2012)]
- Practical calculations for generic orbits in **Schwarzschild**
  [Barack & Sago (2010); Warburton et al. (2012); Osburn et al. (2016)]
- **Gauge-invariant** effects of the conservative self-force
  [Barack & Sago (2009); Shah et al. (2012); Isoyama et al. (2014); . . .]
- Practical calculations for equatorial orbits in **Kerr**
  [van de Meent & Shah (2015); van de Meent (2016)]
- Formulation of **second-order** gravitational self-force
  [Detweiler (2012); Gralla (2012); Pound (2012-2015)]
- Practical calculations at second order
  [Pound (2014); Pound et al. (2016+)]

State of the art

		Adiabatic	1st order	2nd order
Schw.	circular	✔	✔	ongoing
	generic	✔	✔	
Kerr	circular	✔	✔	
	equatorial	✔	✔	
	generic	✔	ongoing	**goal**

Capra Meetings

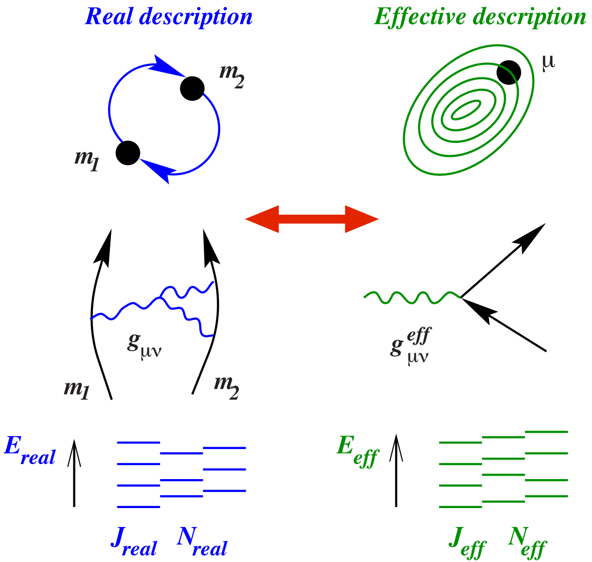
19th Capra Meeting on Radiation Reaction (July 2016, Meudon, France)

Further reading

Review articles

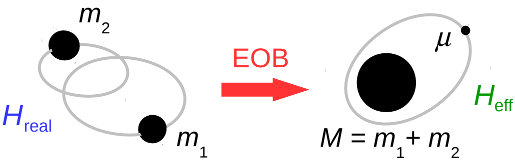
- *Motion of small objects in curved spacetimes*
  A. Pound, in *Equations of Motion in Relativistic Gravity*
  Edited by D. Puetzfeld et al., Springer (2015)
- *The motion of point particles in curved spacetime*
  E. Poisson, A. Pound and I. Vega, Living Rev. Rel. **14**, 7 (2011)
- *Gravitational self force in extreme mass-ratio inspirals*
  L. Barack , Class. Quant. Grav. **26**, 213001 (2009)
- *Analytic black hole perturbation approach to gravitational radiation*
  M. Sasaki and H. Tagoshi, Living Rev. Rel. **6**, 5 (2003)

Outline

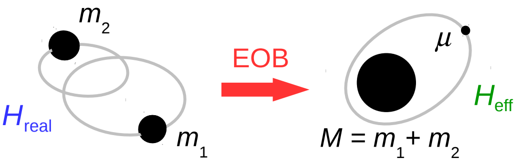


$E_{\rm eff}(J,N) = f(E_{\rm real}(J,N))$

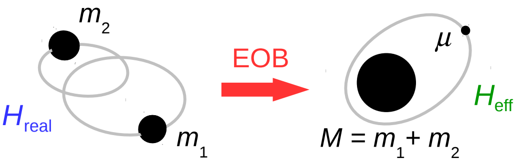
(Credit: Buonanno & Sathyaprakash 2015)



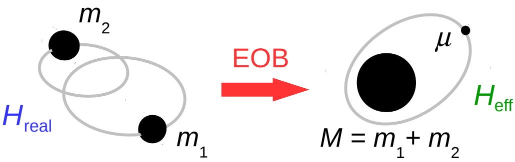
- Motivated by the exact solution in the Newtonian limit



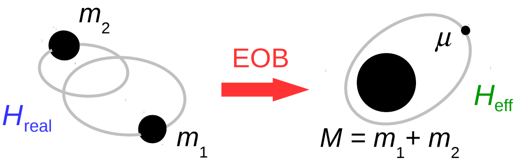
- Motivated by the exact solution in the Newtonian limit
- By *construction*, the EOB model:



- Motivated by the exact solution in the Newtonian limit
- By *construction*, the EOB model:
  - Recovers the known PN dynamics as $c^{-1} \to 0$



- Motivated by the exact solution in the Newtonian limit
- By *construction*, the EOB model:
  - Recovers the known PN dynamics as $c^{-1} \to 0$
  - Recovers the geodesic dynamics when $q \to 0$



- Motivated by the exact solution in the Newtonian limit
- By *construction*, the EOB model:
  - Recovers the known PN dynamics as $c^{-1} \to 0$
  - Recovers the geodesic dynamics when $q \to 0$
- Idea extended to spinning binaries and to tidal effects

EOB Hamiltonian dynamics

EOB Hamiltonian

$$H_{\text{real}}^{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}, \quad \nu \equiv \frac{\mu}{M} \in [0, 1/4]$$

EOB Hamiltonian dynamics

EOB Hamiltonian

$$H_{\mathrm{real}}^{\mathrm{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\mathrm{eff}}}{\mu} - 1\right)}, \quad \nu \equiv \frac{\mu}{M} \in [0, 1/4]$$

Effective Hamiltonian

$$H_{\mathrm{eff}} = \mu\sqrt{g_{tt}^{\mathrm{eff}}(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{g_{rr}^{\mathrm{eff}}(r)} + \cdots\right)}$$

EOB Hamiltonian dynamics

EOB Hamiltonian

$$H_{\text{real}}^{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}, \quad \nu \equiv \frac{\mu}{M} \in [0, 1/4]$$

Effective Hamiltonian

$$H_{\text{eff}} = \mu\sqrt{g_{tt}^{\text{eff}}(r)\left(1 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{g_{rr}^{\text{eff}}(r)} + \cdots\right)}$$

Hamilton's equations

$$\dot{r} = \frac{\partial H_{\text{real}}^{\text{EOB}}}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H_{\text{real}}^{\text{EOB}}}{\partial r} + F_r, \quad \cdots$$

EOB effective metric

Effective metric

$$\mathrm{d}s_{\mathrm{eff}}^2 = -g_{tt}^{\mathrm{eff}}(r;\nu)\,\mathrm{d}t^2 + g_{rr}^{\mathrm{eff}}(r;\nu)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

EOB effective metric

Effective metric

$$\mathrm{d}s_{\text{eff}}^2 = -g_{tt}^{\text{eff}}(r;\nu)\,\mathrm{d}t^2 + g_{rr}^{\text{eff}}(r;\nu)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Effective potentials

$$g_{tt}^{\text{eff}} = \underbrace{1 - \frac{2M}{r}}_{\text{Schwarzschild}} + \underbrace{\nu\left[2\left(\frac{M}{r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\left(\frac{M}{r}\right)^4 + \cdots\right]}_{\text{finite mass-ratio ``deformation''}}$$

EOB effective metric

Effective metric

$$\mathrm{d}s_{\text{eff}}^2 = -g_{tt}^{\text{eff}}(r;\nu)\,\mathrm{d}t^2 + g_{rr}^{\text{eff}}(r;\nu)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Effective potentials

$$g_{tt}^{\text{eff}} = \underbrace{1 - \frac{2M}{r}}_{\text{Schwarzschild}} + \underbrace{\nu\left[2\left(\frac{M}{r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\left(\frac{M}{r}\right)^4 + \cdots\right]}_{\text{finite mass-ratio ``deformation''}}$$

Padé resummation

Motivation: improve convergence of PN series in strong-field regime

EOB waveform generation

Inspiral/plunge

Evolution of Hamiltonian dynamics up to EOB light-ring
Resummations of PN waveform modes and fluxes

$$h^{\text{inspiral}}(t) = \text{``big mess''}$$

EOB waveform generation

Inspiral/plunge

Evolution of Hamiltonian dynamics up to EOB light-ring
Resummations of PN waveform modes and fluxes

$$h^{\text{inspiral}}(t) = \text{“big mess”}$$

Merger/ringdown

Impose continuity with black hole quasinormal modes ringing

$$h^{\text{ringdown}}(t) = \sum_{n\ell m} C_{n\ell m}\, e^{-t/\tau_{n\ell m}} \cos\left(\omega_{n\ell m}(t - t_{\text{m}})\right)$$

EOB waveform generation

Inspiral/plunge

Evolution of Hamiltonian dynamics up to EOB light-ring
Resummations of PN waveform modes and fluxes

$$h^{\text{inspiral}}(t) = \text{``big mess''}$$

Merger/ringdown

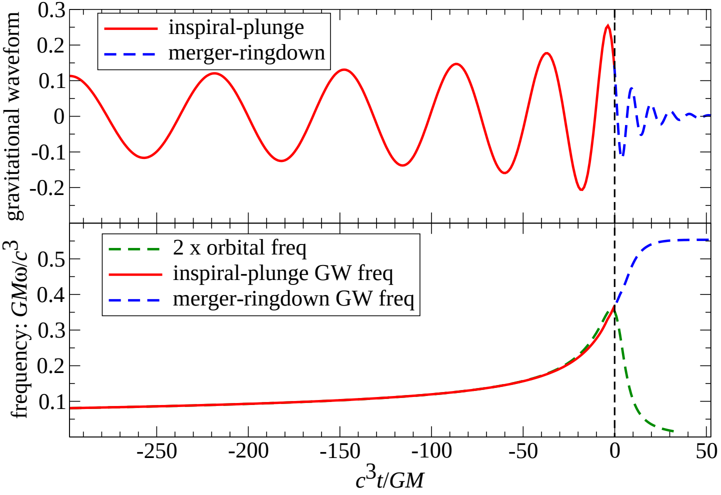
Impose continuity with black hole quasinormal modes ringing

$$h^{\text{ringdown}}(t) = \sum_{n\ell m} C_{n\ell m}\, e^{-t/\tau_{n\ell m}} \cos\left(\omega_{n\ell m}(t - t_{\text{m}})\right)$$

Final EOB waveform

$$h^{\text{EOB}}(t) = \Theta(t_{\text{m}} - t)\, h^{\text{inspiral}}(t) + \Theta(t - t_{\text{m}})\, h^{\text{ringdown}}(t)$$

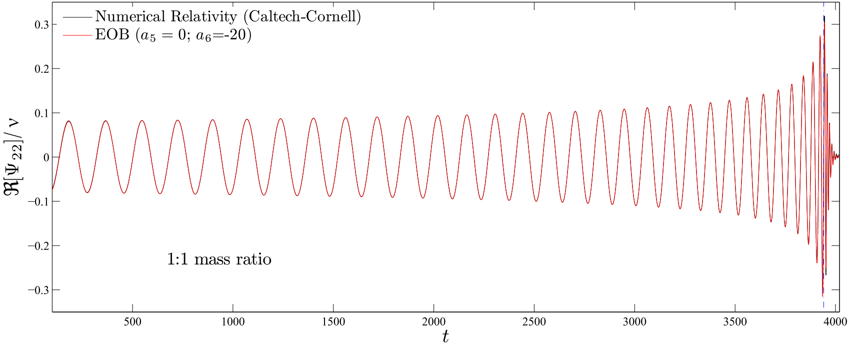
EOB waveform prediction



(Credit: Buonanno & Sathyaprakash 2015)

EOB *vs* NR waveforms

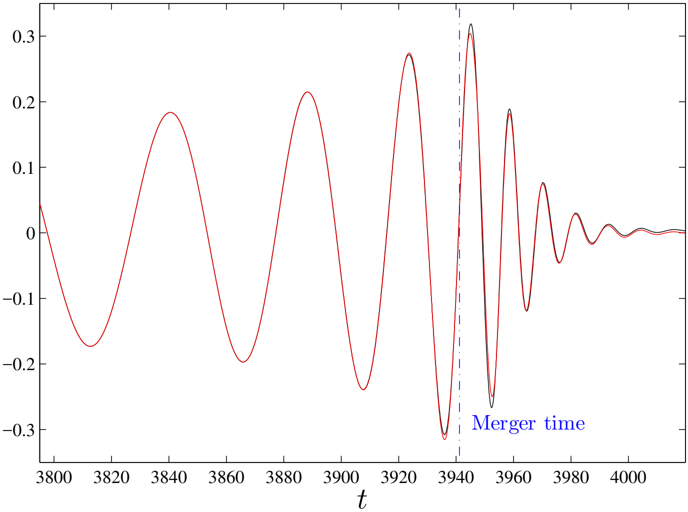
Equal masses and no spins



[PRD **79** (2009) 081503]

EOB *vs* NR waveforms

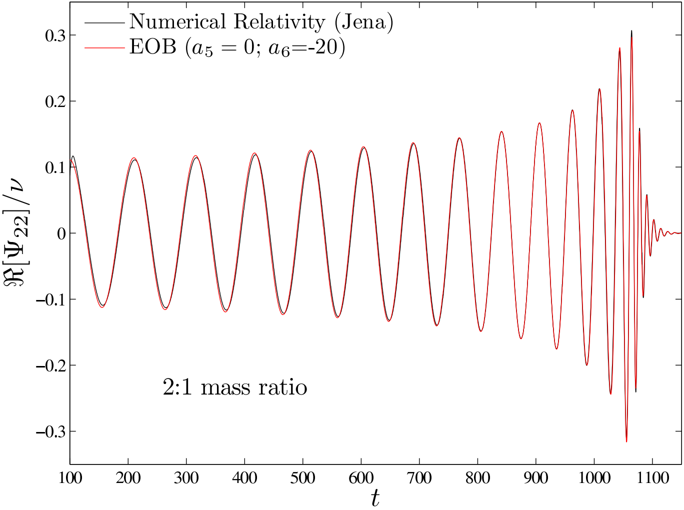
Equal masses and no spins



[PRD **79** (2009) 081503]

EOB *vs* NR waveforms

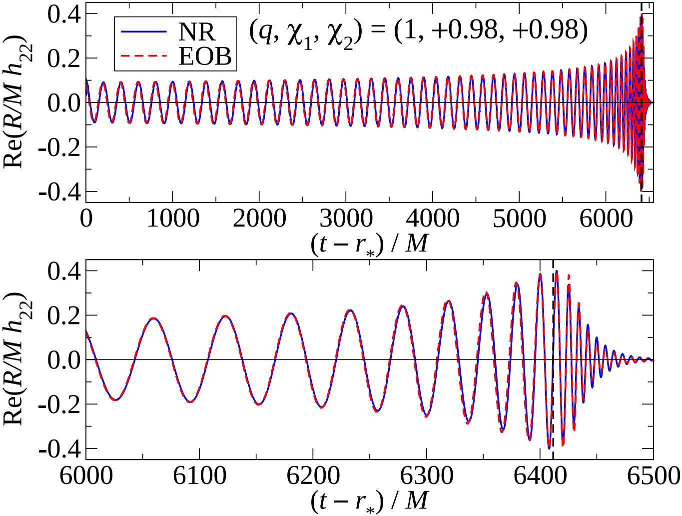
Unequal masses and no spins



[PRD **79** (2009) 081503]

EOB *vs* NR waveforms

Equal masses and aligned spins

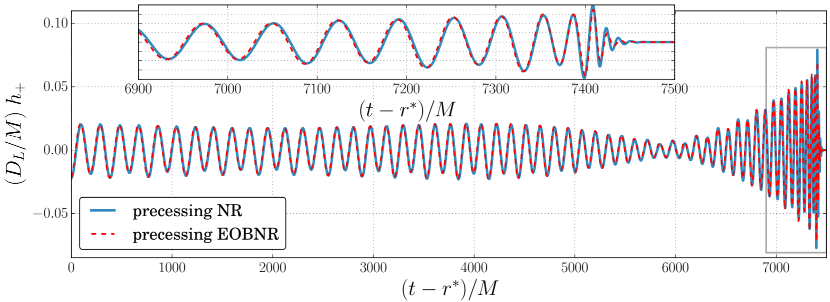


[PRD **89** (2014) 061502]

EOB *vs* NR waveforms

Unequal masses and a precessing spin

$$(q, \chi_1, \chi_2) = (5, +0.5, 0),\ \iota = \pi/3$$



[gr-qc/1607.05661]

Recent developments

- Extension of EOB model to **spinning** binaries

  [Barausse & Buonanno (2011), Nagar (2011), Damour & Nagar (2014)]

Recent developments

- Extension of EOB model to **spinning** binaries
  [Barausse & Buonanno (2011), Nagar (2011), Damour & Nagar (2014)]
- Addition of **tidal** interactions for neutrons stars
  [Damour & Nagar (2010), Bini et al. (2012), Hinderer et al. (2016)]

Recent developments

- Extension of EOB model to **spinning** binaries
  [Barausse & Buonanno (2011), Nagar (2011), Damour & Nagar (2014)]
- Addition of **tidal** interactions for neutrons stars
  [Damour & Nagar (2010), Bini et al. (2012), Hinderer et al. (2016)]
- Various **calibrations** to numerical relativity simulations
  [Damour & Nagar (2014), Pan et al. (2014), Taracchini et al. (2014)]

Recent developments

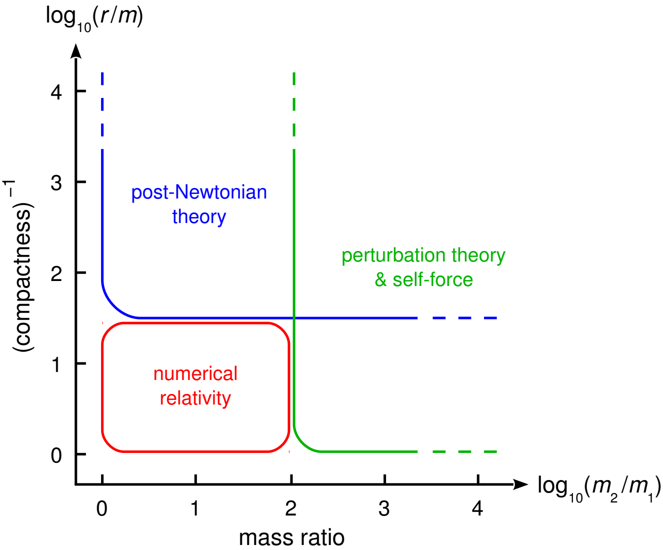
- Extension of EOB model to **spinning** binaries
  [Barausse & Buonanno (2011), Nagar (2011), Damour & Nagar (2014)]
- Addition of **tidal** interactions for neutrons stars
  [Damour & Nagar (2010), Bini et al. (2012), Hinderer et al. (2016)]
- Various **calibrations** to numerical relativity simulations
  [Damour & Nagar (2014), Pan et al. (2014), Taracchini et al. (2014)]
- Calibration of EOB potentials by comparison to **self-force**
  [Barack et al. (2011), Le Tiec (2015), Akcay & van de Meent (2016)]

Further reading

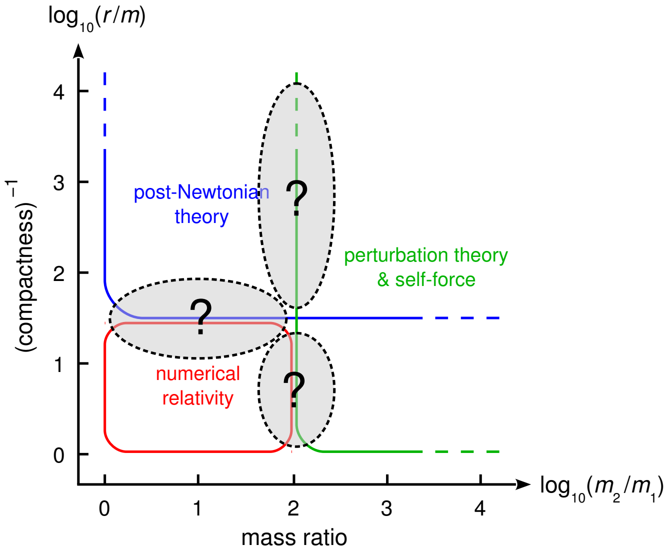
Review articles

- *Sources of gravitational waves: Theory and observations*
  A. Buonanno and B. S. Sathyaprakash, in *General relativity and gravitation: A centennial perspective*
  Edited by A. Ashtekar et al., Cambridge University Press (2015)
- *The general relativistic two body problem and the EOB formalism*
  T. Damour, in *General relativity, cosmology and astrophysics*
  Edited by J. Bicák and T. Ledvinka, Springer (2014)
- *The effective one-body description of the two-body problem*
  T. Damour and A. Nagar, in *Mass and motion in general relativity*
  Edited by L. Blanchet et al., Springer (2011)

Outline



$\log_{10}(r/m)$
4
3
2
1
0
$(\text{compactness})^{-1}$
post-Newtonian theory
perturbation theory & self-force
numerical relativity
0
1
2
3
4
$\log_{10}(m_2/m_1)$
mass ratio



$\log_{10}(r/m)$
$(\text{compactness})^{-1}$
4
3
2
1
0
post-Newtonian
theory
?
?
perturbation theory
& self-force
numerical
relativity
?
0
1
2
3
4
$\log_{10}(m_2/m_1)$
mass ratio

Why?

- Independent checks of long and complicated calculations
- Identify domains of validity of approximation schemes
- Extract information inaccessible to other methods
- Develop a universal model for compact binaries

Why?

- Independent checks of long and complicated calculations
- Identify domains of validity of approximation schemes
- Extract information inaccessible to other methods
- Develop a universal model for compact binaries

How?

✘ Use the same coordinate system in all calculations

✔ Using coordinate-invariant relationships

Why?

- Independent checks of long and complicated calculations
- Identify domains of validity of approximation schemes
- Extract information inaccessible to other methods
- Develop a universal model for compact binaries

How?

✘ Use the same coordinate system in all calculations

✔ Using coordinate-invariant relationships

What?

- Gravitational waveforms at future null infinity
- Conservative effects on the orbital dynamics

Paper	Year	Methods	Observable	Orbit	Spin
Baker *et al.*	2007	NR/PN	waveform		
Boyle *et al.*	2007	NR/PN	waveform		
Hannam *et al.*	2007	NR/PN	waveform		
Boyle *et al.*	2008	NR/PN/EOB	energy flux		
Damour & Nagar	2008	NR/EOB	waveform		
Hannam *et al.*	2008	NR/PN	waveform		✓
Pan *et al.*	2008	NR/PN/EOB	waveform		
Campanelli *et al.*	2009	NR/PN	waveform		✓
Hannam *et al.*	2010	NR/PN	waveform		✓
Hinder *et al.*	2010	NR/PN	waveform	eccentric	
Lousto *et al.*	2010	NR/BHP	waveform		
Sperhake *et al.*	2011	NR/PN	waveform		
Sperhake *et al.*	2011	NR/BHP	waveform	head-on	
Lousto & Zlochower	2011	NR/BHP	waveform		
Nakano *et al.*	2011	NR/BHP	waveform		
Lousto & Zlochower	2013	NR/PN	waveform		
Nagar	2013	NR/BHP	recoil velocity		
Hinder *et al.*	2014	NR/PN/EOB	waveform		✓
Szilagyi *et al.*	2015	NR/PN/EOB	waveform		
Ossokine *et al.*	2015	NR/PN	waveform		✓

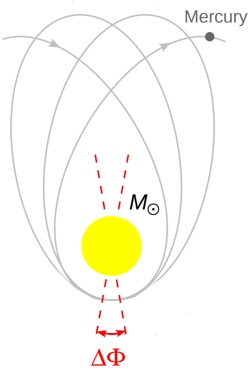
Paper	Year	Methods	Observable	Orbit	Spin
Detweiler	2008	BHP/PN	redshift observable		
Blanchet *et al.*	2010	BHP/PN	redshift observable		
Damour	2010	BHP/EOB	ISCO frequency		
Mroué *et al.*	2010	NR/PN	periastron advance		
Barack *et al.*	2010	BHP/EOB	periastron advance		
Favata	2011	BHP/PN/EOB	ISCO frequency		
Le Tiec *et al.*	2011	NR/BHP/PN/EOB	periastron advance		
Damour *et al.*	2012	NR/EOB	binding energy		
Le Tiec *et al.*	2012	NR/BHP/PN/EOB	binding energy		
Akcay *et al.*	2012	BHP/EOB	redshift observable		
Hinderer *et al.*	2013	NR/EOB	periastron advance		✓
Le Tiec *et al.*	2013	NR/BHP/PN	periastron advance		✓
Bini & Damour Shah *et al.* Blanchet *et al.*	2014	BHP/PN	redshift observable		
Dolan *et al.* Bini & Damour	2014	BHP/PN	precession angle		✓
Isoyama *et al.*	2014	BHP/PN/EOB	ISCO frequency		✓
Akcay *et al.*	2015	BHP/PN	averaged redshift	eccentric	
Shah & Pound	2015	BHP/PN	precession angle		✓
Zimmerman *et al.*	2016	NR/PN	surface gravity		
Akcay *et al.*	2016	BHP/PN	precession angle	eccentric	

Relativistic perihelion advance of Mercury

- Observed anomalous advance of Mercury's perihelion of $\sim$ 43" /cent.
- Accounted for by the leading-order relativistic angular advance per orbit

$$\Delta\Phi = \frac{6\pi G M_\odot}{c^2 a\,(1 - e^2)}$$

- Periastron advance of $\sim 4^\circ$/yr now measured in binary pulsars



Periastron advance in black hole binaries

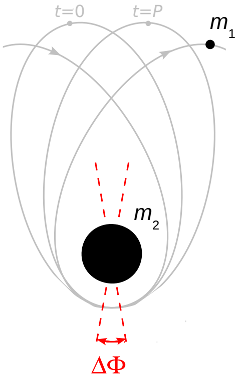
- Generic eccentric orbit parametrized by the two invariant frequencies

$$\Omega_r = \frac{2\pi}{P}\,, \quad \Omega_\varphi = \frac{1}{P}\int_0^P \dot{\varphi}(t)\,\mathrm{d}t$$

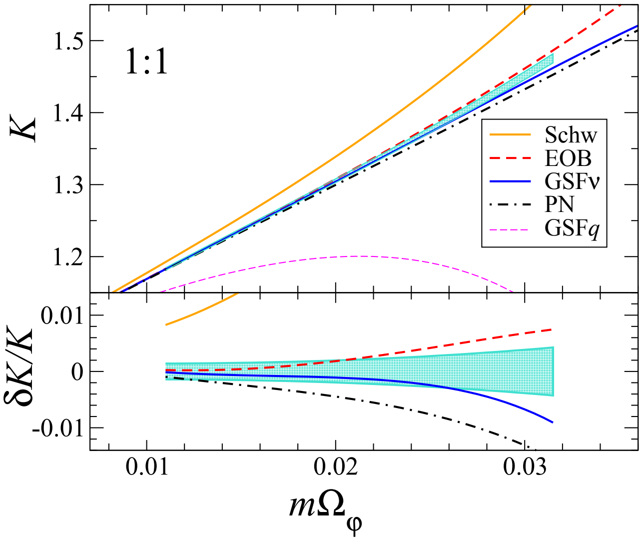
- Periastron advance per radial period

$$K \equiv \frac{\Omega_\varphi}{\Omega_r} = 1 + \frac{\Delta\Phi}{2\pi}$$

- In the circular orbit limit $e \to 0$, the relation $K(\Omega_\varphi)$ is coordinate-invariant

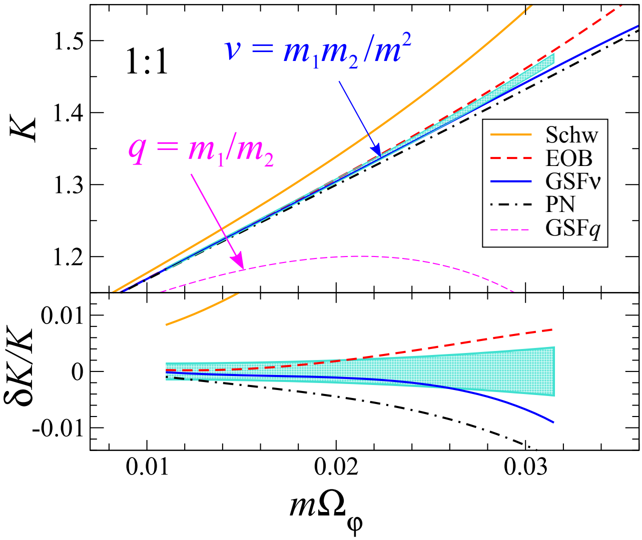


Periastron advance *vs* orbital frequency



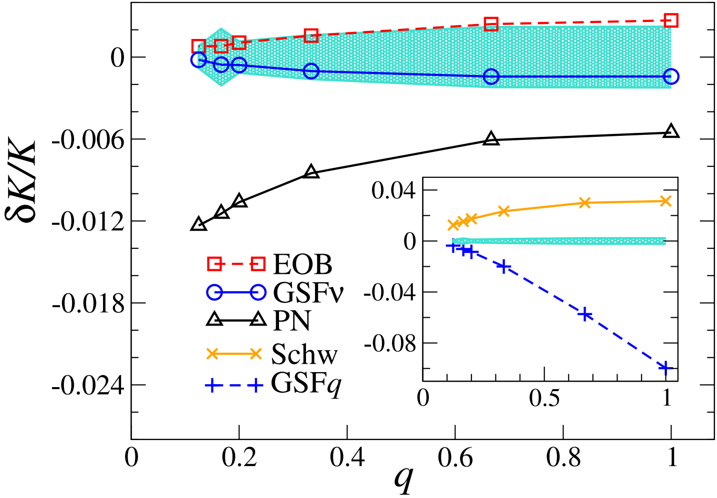
[PRL **107** (2011) 141101]

Periastron advance *vs* orbital frequency



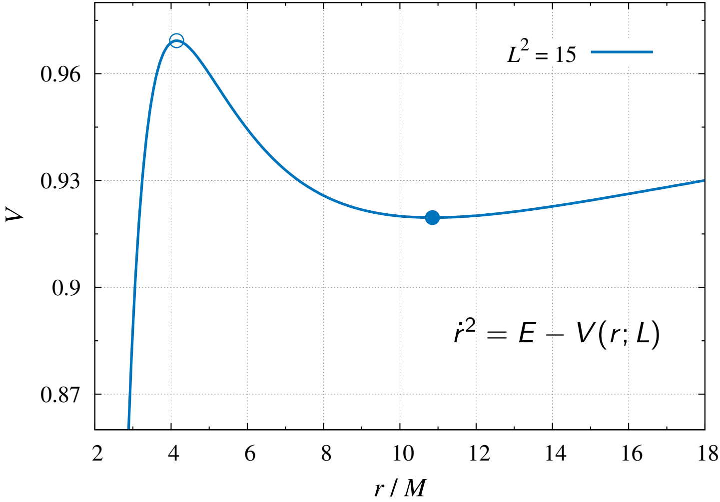
[PRL **107** (2011) 141101]

Periastron advance *vs* mass ratio

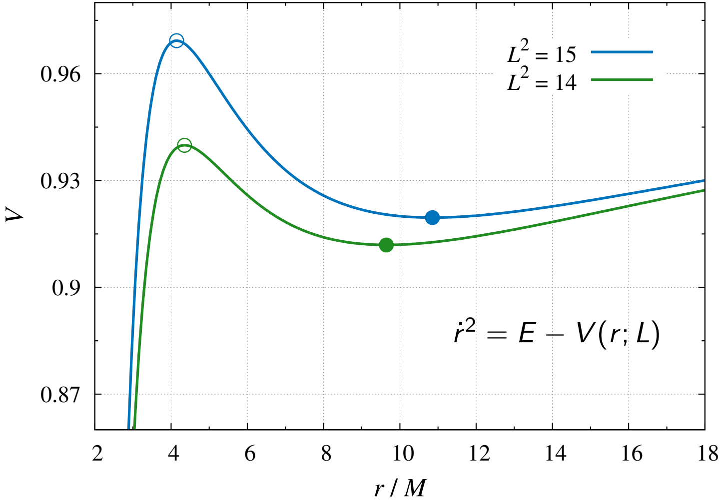


[PRL **107** (2011) 141101]

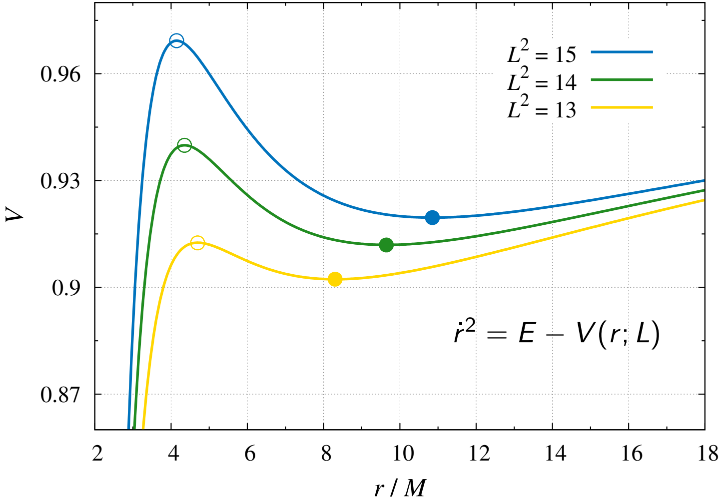
Innermost stable circular orbit (ISCO)



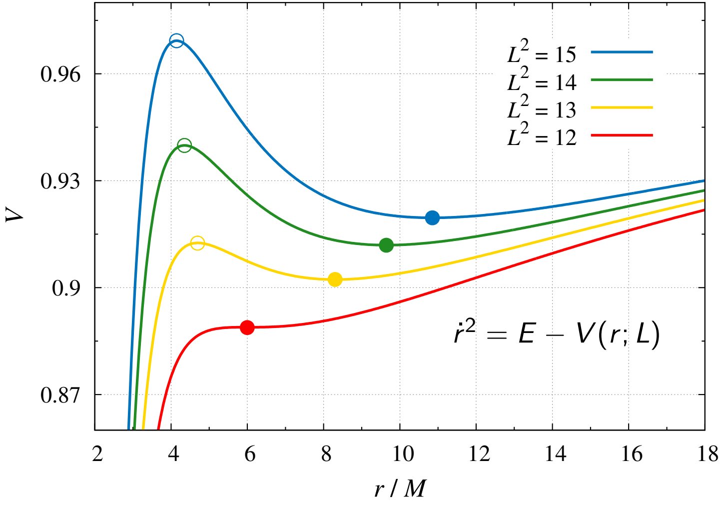
Innermost stable circular orbit (ISCO)



Innermost stable circular orbit (ISCO)



Innermost stable circular orbit (ISCO)



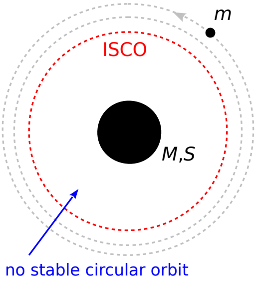
Innermost stable circular orbit (ISCO)

- The innermost stable circular orbit is identified by a vanishing restoring radial force under small-e perturbations:

$$\frac{\partial^2 H}{\partial r^2} = 0 \quad \longrightarrow \quad \Omega_{\mathrm{ISCO}}$$

- The minimum energy circular orbit is the most bound orbit along a sequence of circular orbits:

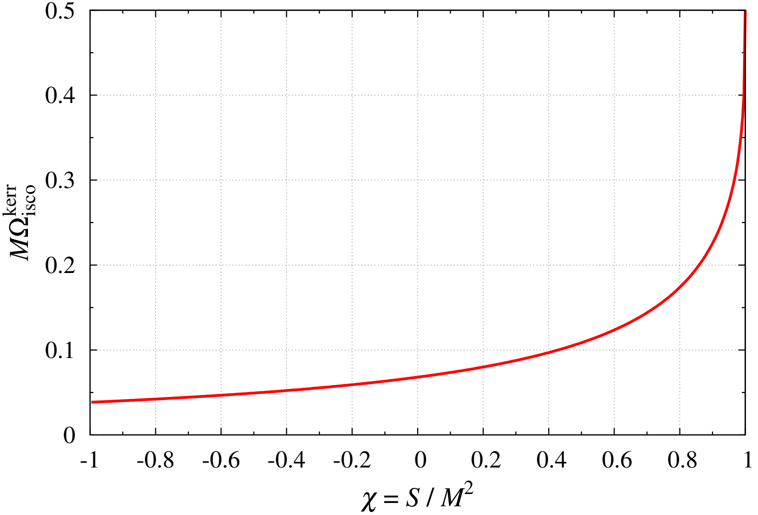
$$\frac{\partial E}{\partial \Omega} = 0 \quad \longrightarrow \quad \Omega_{\mathrm{MECO}}$$



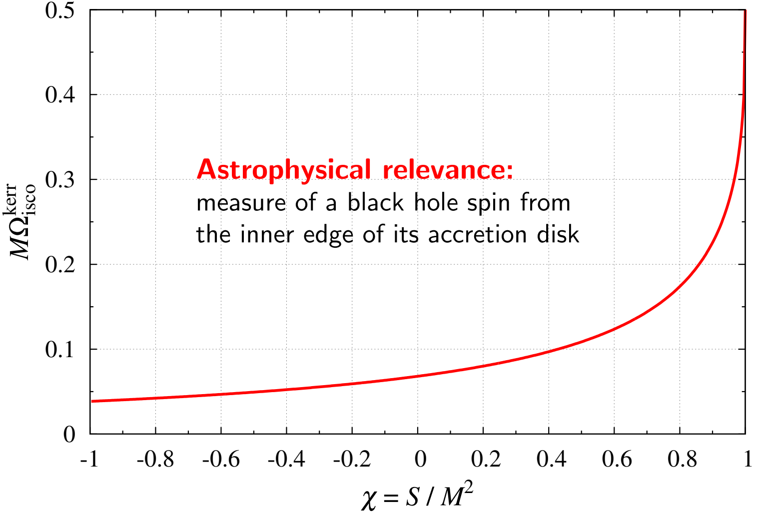
- For Hamiltonian systems [Buonanno *et al.* (2003)]

$$\Omega_{\mathrm{ISCO}} = \Omega_{\mathrm{MECO}}$$

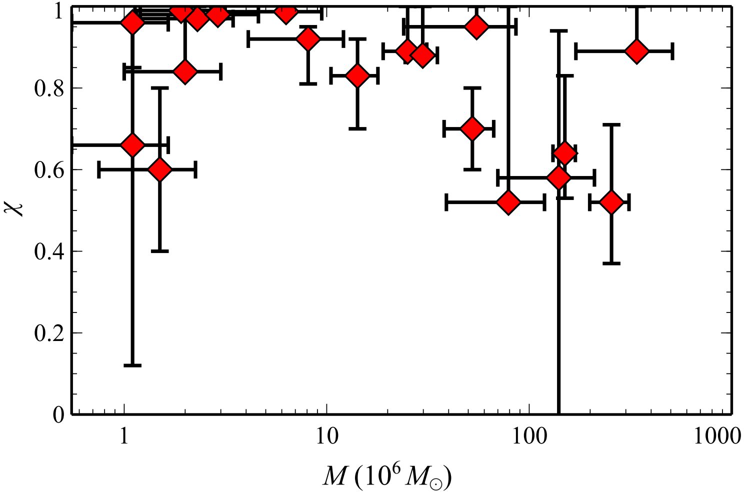
Kerr ISCO frequency *vs* black hole spin



Kerr ISCO frequency *vs* black hole spin



Spins of supermassive black holes



[CQG **30** (2013) 244004]

Frequency shift of the Kerr ISCO

- The orbital frequency of the Kerr ISCO is shifted under the effect of the conservative self-force:

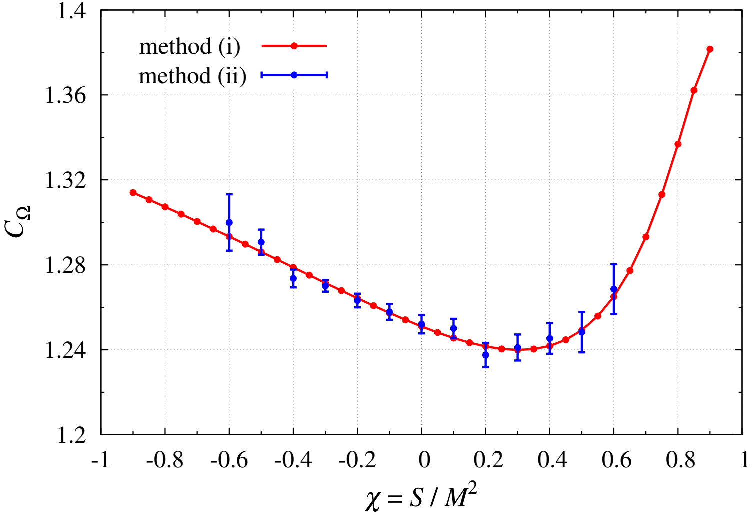
$$(M+m)\Omega_{\text{isco}} = \underbrace{M\Omega^{\text{kerr}}_{\text{isco}}(\chi)}_{\substack{\text{test mass}\\ \text{result}}} \left\{ 1 + \underbrace{q\, C_\Omega(\chi)}_{\substack{\text{conservative}\\ \text{GSF effect}}} + \mathcal{O}(q^2) \right\}$$

- The frequency shift can be computed from a stability analysis of slightly eccentric orbits near the Kerr ISCO
- Combining the Hamiltonian first law with the MECO conditio $\partial E/\partial\Omega = 0$ yields the same result:

$$C_\Omega = \frac{1}{2}\,\frac{z''_{\text{GSF}}(\Omega^{\text{kerr}}_{\text{isco}})}{E''(\Omega^{\text{kerr}}_{\text{isco}})}$$

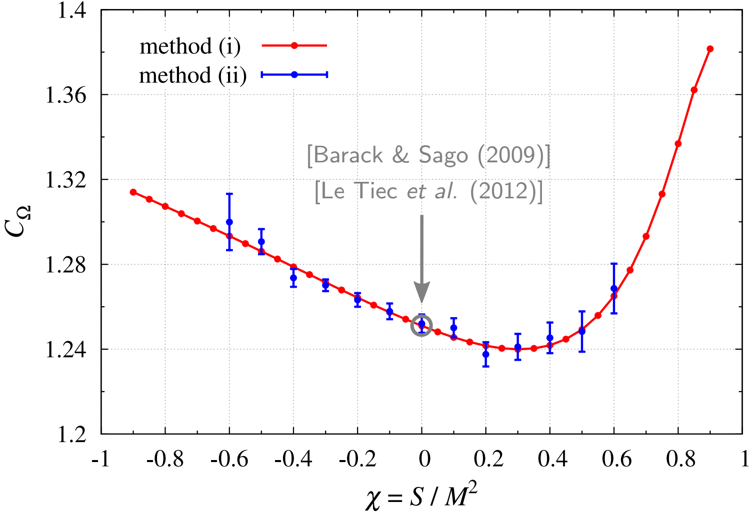
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



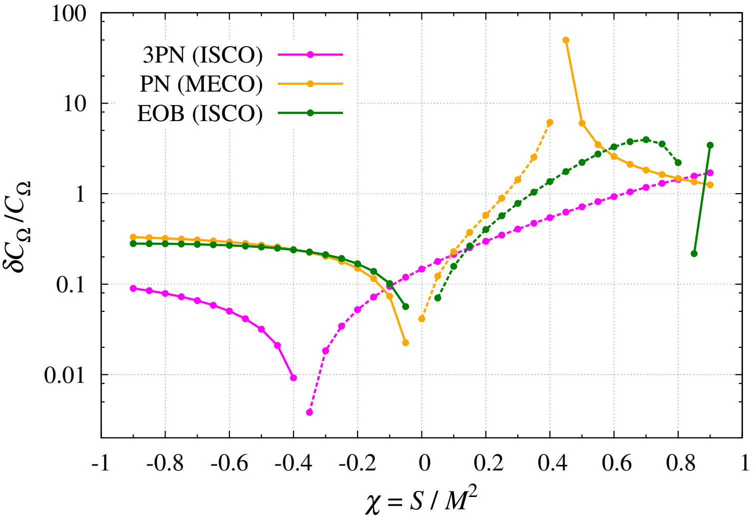
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



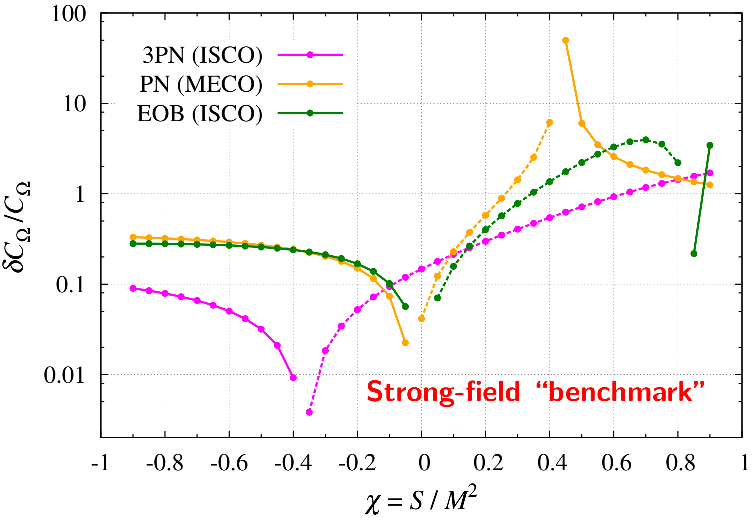
[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



[PRL **113** (2014) 161101]

ISCO frequency shift *vs* black hole spin



[PRL **113** (2014) 161101]