Gravitational wave source modelling: fundamental results

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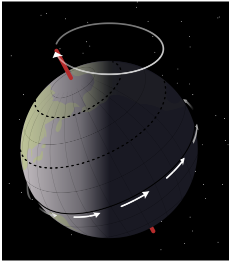
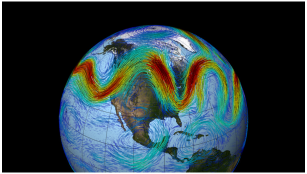
Context: emission mechanisms

- Interested in gravitational wave emission from *individual* neutron stars.
- Three possible emission mechanisms:

Mountains



Free precession

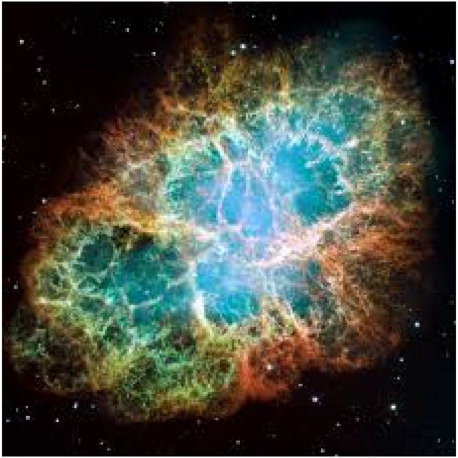


Fluid oscillations

- I'll describe first two; Kostas G will describe the last.
- MAP will discuss the (many!) signal analysis issues.

Targeted searches

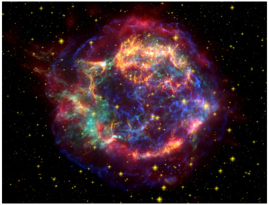
- Look for signals from known neutron stars, e.g. radio pulsars.



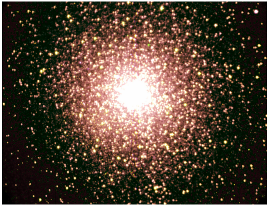
Crab nebula

Directed searches

- Searches over small sky regions, but no known timing solution, e.g.



Cas A

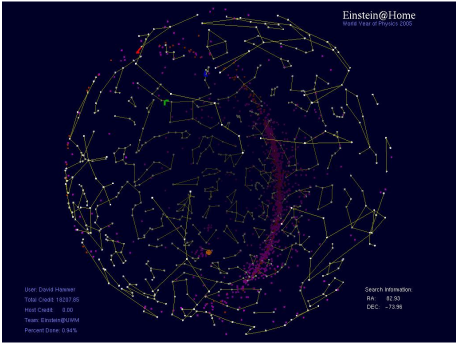


Globular cluster 47 Tuc

Galactic centre

All-sky searches

- Search over all directions, all frequencies.
- Some searches make use of Einstein@Home.



E@H screen shot

Outline

- Basic gravitational wave formulae.
- Application to steadily rotating stars.
- Spin-down upper limits.

Two dimensionless numbers

Just how 'relativistic' are neutron stars? Look at two dimensionless numbers:

1. Compactness (measure of importance of GR):

$$\frac{M}{R} = 0.21 \left(\frac{M}{1.4\, M_\odot} \right) \left(\frac{10\,\mathrm{km}}{R} \right). \tag{1}$$

2. Equatorial speed:

$$\frac{v}{c} = \frac{2\pi R \nu}{c} \approx 0.15 \left(\frac{R}{10\,\mathrm{km}} \right) \left(\frac{\nu}{716\,\mathrm{Hz}} \right), \tag{2}$$

   for fastest observed pulsar, PSR J1748-2446ad, spin frequency $\nu = 716$ Hz.

This motivates treating GW emission as a *weak-field, slow motion correction to Newtonian theory.*

The slow-motion, weak-field formalism

- Classic reference is Thorne (1980), where extensive use is made of spherical harmonics and symmetric trace-free tensors.
- Key results are given in terms of the (complex) mass and mass current multipoles:

$$I^{lm} = A_l \int \rho Y^*_{lm} r^l \, dV, \qquad S^{lm} = \int \rho v_j Y_j^{B,\, lm\, *} r^l dV, \tag{3}$$

  where A_l and B_l are l-dependent prefactors, Y_{lm} a spherical harmonic, and $Y_j^{B,\, lm} = -[l(l+1)]^{-1/2} \mathbf{r} \times \nabla Y^{lm}$, a vector spherical harmonic.

- GW field is given by:

$$h_{jk}^{\mathrm{TT}} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[\frac{1}{r} {}^{(l)} I^{lm}(t-r) T_{jk}^{E2,lm} + \frac{1}{r} {}^{(l)} S^{lm}(t-r) T_{jk}^{B2,lm} \right] \tag{4}$$

  where $T_{jk}^{E2,lm}$ and $T_{jk}^{B2,lm}$ are tensor spherical harmonics, and pre-superscript a time derivative.

- GW luminosity is given by:

$$\dot{E} = \frac{1}{32\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} < |{}^{(l+1)} I^{lm}|^2 + |{}^{(l+1)} S^{lm}|^2 > \tag{5}$$

The mass quadrupole formalism

- Each successive multipole in h_{jk}^{TT} is factor v/c smaller than the last.
- $\Rightarrow$ normally only consider $l = 2$ case, the *quadrupole*.
- In most situation, mass quadrupole dominates over mass current quadrupole; exception is the r-mode oscillation (see talk by Glampedakis).
- Key *mass quadrupole* equations are then:

$$I_{2m} = \frac{16\pi\sqrt{3}}{15} \int \rho Y_{2m}^{*} r^2 \, dV. \tag{6}$$

$$h_{ab}^{\mathrm{TT}}(t) = \frac{1}{r} \sum_m \ddot{I}^{2m} T_{ab}^{\mathrm{E}2,2m}. \tag{7}$$

$$\frac{dE}{dt} = \frac{1}{32\pi} \sum_m < |^{(3)} I_{2m}|^2 > . \tag{8}$$

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Alternative formalism: quadrupole moment tensor

- Can be more convenient to work with the mass quadrupole moment tensor:

$$I_{ab} = \int \rho x_a x_b \, dV. \tag{9}$$

- It is often the symmetric and trace-free (STF) part of this that appears in the GW equations:

$$\mathcal{I}_{ab} = [I_{ab}]^{\mathrm{STF}} = \int \rho (x_a x_b - \frac{1}{3}\delta_{ab} r^2) \, dV = I_{ab} - \frac{1}{3}\delta_{ab} I_{cc}. \tag{10}$$

- Expression for the GW luminosity is then very simple:

$$\dot{E} = \frac{1}{5} < \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}_{ij} > . \tag{11}$$

Quadrupole moment tensor cont...

- Can also calculate wave field:

$$\bar{h}^{\mathrm{TT}}_{ab}(t, \mathbf{x}) = \frac{2}{r} \ddot{\mathcal{I}}^{\mathrm{TT}}_{ab}(t - r). \tag{12}$$

- Have made use of *projection operator* P_{ab}:

$$P_{ab} = \delta_{ab} - n_a n_b, \tag{13}$$

for projection into plane orthogonal to unit 3-vector n_a.

- Then, in matrix notation (see MTW, Box 35.1):

$$\mathcal{I}^{\mathrm{TT}} = P\mathcal{I}P - \frac{1}{2} P \,\mathrm{Tr}(P\mathcal{I}). \tag{14}$$

Relation with moment of inertia tensor

- For rigidly rotating bodies, there exists a simple relationship between the angular momentum J_a and the angular velocity Ω_a.
- For single particle, $\mathbf{J} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}$.
- For extended body:

$$\mathbf{J} = \int \rho \mathbf{r} \times \mathbf{v} \, dV. \tag{15}$$

- For rigid rotation, $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$:

$$\mathbf{J} = \int \rho \mathbf{r} \times (\mathbf{\Omega} \times \mathbf{r}) \, dV. \tag{16}$$

- Can manipulate to show:

$$J_a = I^{\mathrm{MoI}}_{ab} \Omega^b, \qquad I^{\mathrm{MoI}}_{ab} \equiv \int \rho(\delta_{ab} x^x x_c - x_a x_b) \, dV. \tag{17}$$

Relation with moment of inertia tensor cont...

- $I^{\rm MoI}_{ab}$ is a symmetric second rank matrix, so can be diagonalised by an orthogonal rotation, so that in 'body frame'

$$I^{\rm MoI}_{ab} = \begin{bmatrix} I^{\rm MoI}_{xx} & & \\ & I^{\rm MoI}_{yy} & \\ & & I^{\rm MoI}_{zz} \end{bmatrix}. \tag{18}$$

- These components are the *moments of inertia* about the *principal* or *body axes*, e.g. about the z-axis:

$$I^{\rm MoI}_{zz} = \int \rho(x^2 + y^2)\, dV. \tag{19}$$

- Comparing with quadrupole moment tensor

$$I_{ab} = \int \rho x_a x_b \, dV, \tag{20}$$

we see that

$$I^{\rm MoI}_{ab} = \delta_{ab} \int \rho x^c x_c \, dV - I_{ab}. \tag{21}$$

- It follows that *differences* in diagonal components are related by an overall sign, e.g.

$$I_{xx} - I_{yy} = -(I^{\rm MoI}_{yy} - I^{\rm MoI}_{xx}). \tag{22}$$

Computing I_{ab} for a rotating star

- In rotating body frame:

$$I_{ab} = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}. \tag{23}$$

- Want components I_{ab} with respect to inertial frame.
- Use an active time-dependent rotation through angle ϕ about Oz, e.g. for a vector v^a, the components of the unrotated vector $v^a(0)$ and the components of the rotated vector $v^a(\phi)$ are related by

$$v^a(\phi) = R^a{}_b v^b(0), \qquad \text{where} \qquad R^a{}_b = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

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Computing I_{ab} for a rotating star ...

- For I_{ab}, transformation becomes

$$I_{ab}(\phi) = R_a{}^c R_b{}^d I_{cd}(0) = R_a{}^c I_{cd}(0) (R^{\mathrm{T}})^d{}_b = [R I(0) R^{\mathrm{T}}]_{ab}. \tag{25}$$

- Carrying out the algebra, get:

$$I_{ab}(\phi) = \frac{1}{2}(I_{xx} - I_{yy}) \begin{bmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_{xx} + I_{yy} & 0 & 0 \\ & I_{xx} + I_{yy} & 0 \\ 0 & 0 & 2I_{zz} \end{bmatrix} \tag{26}$$

- Second term independent of ϕ, so is constant in time. First term has zero trace, so

$$\dot{\mathcal{I}}_{ab}(\phi) = \dot{I}_{ab}(\phi), \tag{27}$$

and similarly for all higher time derivatives.

Gravitational wave luminosity

- Luminosity given by

$$L = \frac{1}{5} < \dddot{\mathcal{I}}_{ab} \dddot{\mathcal{I}}_{ab} > . \tag{28}$$

- Using calculated I_{ab} find

$$L = \frac{32}{5} \Omega^6 (I_{xx} - I_{yy})^2. \tag{29}$$

- Define *ellipticity*:

$$\epsilon \equiv \frac{I_{xx} - I_{yy}}{I_{zz}^{\mathrm{MoI}}} = \frac{I_{yy}^{\mathrm{MoI}} - I_{xx}^{\mathrm{MoI}}}{I_{zz}^{\mathrm{MoI}}}. \tag{30}$$

- Then

$$L = \frac{32}{5} \Omega^6 (I_{zz}^{\mathrm{MoI}} \epsilon)^2. \tag{31}$$

Spin-down upper limits

- For a star with an observed spin frequency Ω and spin-down rate $\dot{\Omega}$, can get an upper limit on ellipticity by assuming that all spin-down is due to GW emission:

$$\frac{d}{dt}\left(\frac{1}{2} I_{zz}^{\mathrm{MoI}} \Omega^2\right) = -\frac{32}{5}\Omega^6 (I_{zz}^{\mathrm{MoI}} \epsilon)^2. \tag{32}$$

- Rearranging, and writing in terms of spin period $P = 2\pi/\Omega$:

$$\epsilon_{\mathrm{spindown}} = \left[\frac{5\dot{P}P^3}{32(2\pi)^4 I_{zz}^{\mathrm{MoI}}}\right]^{1/2}. \tag{33}$$

- Example: For Crab pulsar, $P \approx 0.0334$ s, $\dot{P} \approx 4.21 \times 10^{-13}$ s s^{-1} (and $I_{zz}^{\mathrm{MoI}} \approx 10^{45}$ g cm^2 for all neutron stars), giving

$$\epsilon_{\mathrm{spindown}}(\mathrm{Crab}) \approx 7.6 \times 10^{-4}. \tag{34}$$

The gravitational wave field

- Now calculate GW field using

$$h^{\mathrm{TT}}_{ab} = \frac{2}{r}\ddot{\mathcal{I}}^{\mathrm{TT}}_{ab}, \qquad \ddot{\mathcal{I}}^{\mathrm{TT}}_{ab} = P\ddot{\mathcal{I}}_{ab}P - \frac{1}{2}P_{ab}\,\mathrm{Tr}(P\mathcal{I}), \tag{35}$$

  where P is the projection operator $P_{ab} = \delta_{ab} - n_a n_b$ for GWs propagating in direction of unit vector n_a.

- Set $n_a = (0, 0, 1)$ for GWs emitted along Oz, i.e. along rotation axis:

$$h^{\mathrm{TT}}_{ab}(Oz) = \frac{4}{r}\Omega^2 I^{\mathrm{MoI}}_{zz}\epsilon \left\{ \sin 2\phi \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cos 2\phi \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \tag{36}$$

- Clearly we have $h_+ = h_\times$, i.e. circular polarisation, motivating definition of h_0:

$$h_0 \equiv \frac{4}{r}\Omega^2 I^{\mathrm{MoI}}_{zz}|\epsilon|. \tag{37}$$

Gravitational wave amplitudes

- Inserting numbers:

$$h_0 = 1.05 \times 10^{-27} \left(\frac{10\,\mathrm{kpc}}{r} \right) \left(\frac{f_{\mathrm{GW}}}{100\,\mathrm{Hz}} \right)^2 \left(\frac{I_{zz}^{\mathrm{MoI}}}{10^{45}\,\mathrm{g\,cm^2}} \right) \left(\frac{\epsilon}{10^{-6}} \right). \tag{38}$$

- For stars with measured P and $\dot{P}$, can set $\epsilon = \epsilon_{\mathrm{spindown}}$ to obtain a spindown upper limit on h_0.
- Example: For Crab pulsar we had $\epsilon_{\mathrm{spindown}} \approx 7.6 \times 10^{-4}$, $f_{\mathrm{GW}} \approx 60$ Hz, and astronomers estimate $r \approx 2$ kpc, giving

$$h_0^{\mathrm{spindown}}(\mathrm{Crab}) \approx 1.4 \times 10^{-24}. \tag{39}$$

- GW amplitudes clearly small, but can try to detect signals by coherently combining long data stretches (months–years); MAP's lecture.

Upper limits: spin down and direct

- Can plot spin down upper limit and actual ‘direct’ upper limit on the same diagram.
- Dimensionless noise curves fold-in duration of observation run, noise $\sim [S_h(f)/T_{\rm obs}]^{1/2}$ (see MAP's lecture).
- Non-detection of Crab by LIGO in fact shows that mountain is smaller than this; current limit is $\epsilon \lesssim 9 \times 10^{-5}$; no more than 1.2% of spin-down energy going into GWs!

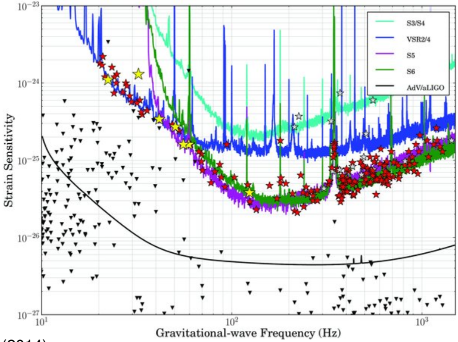
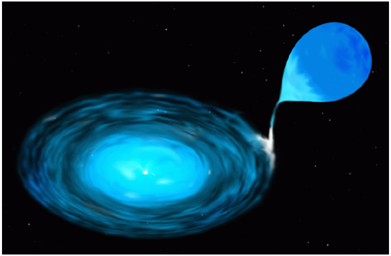


Figure: Aasi et al Ap.J. **785** 119 (2014).

Triaxial stars: upper limits for accreting systems



- Can estimate simple upper bounds on GW emission from accreting triaxial star too.
- Balance spin-up accretion torque against spin-down gravitational torque:

$$\dot{M}(2Mr_{\rm acc})^{1/2} \sim (\epsilon M R^2 \Omega^3)^2/\Omega \tag{40}$$

for angular momentum deposition at radius $r_{\rm acc}$.

- Can estimate accretion rate $\dot{M}$ from observed X-ray flux.
- Again, knowing distance, can bound h; see Bildsten (1998), Haskell+ (2015).

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Summary

- We summarised the key results for GW emission from nearly-Newtonian sources.
- The emitted GW signal from a mountain is proportional to the ellipticity, and the square of its rotation frequency.
- Can use observed spin-down rate to put upper bounds on ellipticity.
- Not clear how close to these upper bounds the real ellipticities will be.
- What determines actual ellipticities? Wait for my **next lecture!**

Exercises

1. Verify the formulae given for compactness and equatorial velocity of equations (1) and (2).
2. Verify that equation (17) for the moment of inertia really does follow from equation (16).
3. Convince yourself that the rotation matrix of equation (24) is of the correct form.
4. Verify equation (26), giving the quadrupole moment of a rotating star, as expressed in the inertial frame, and hence verify the GW luminosity of equation (29).
5. Verify equation (36) for the TT-gravitational wave field of a rotating star, emitted along the rotation axis. How does the result change for emission along an axis lying in the equatorial plane? What about emission is some arbitrary direction?
6. (Most important!) Which of these slides was inserted at the last moment?