General relativity computations with SageManifolds

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Outline

1. Computer differential geometry and tensor calculus
2. The SageManifolds project
3. Let us practice!
4. Other examples
5. Conclusion and perspectives

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- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]

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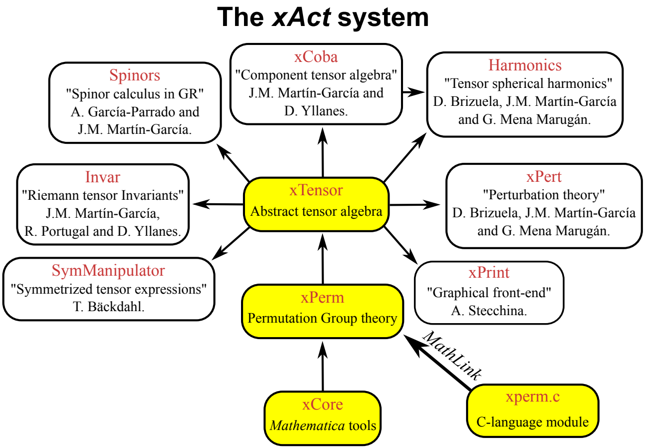
- Since then, many softwares for tensor calculus have been developed...

An example of modern software: The xAct suite

Free packages for tensor computer algebra in Mathematica, developed by José Martín-García et al. http://www.xact.es/



[García-Parrado Gómez-Lobo & Martín-García, Comp. Phys. Comm. **183**, 2214 (2012)]

Software for differential geometry

Packages for general purpose computer algebra systems:

- xAct free package for Mathematica [J.-M. Martin-Garcia]
- Ricci free package for Mathematica [J. L. Lee]
- MathTensor package for Mathematica [S. M. Christensen & L. Parker]
- DifferentialGeometry included in Maple [I. M. Anderson & E. S. Cheb-Terrab]
- Atlas 2 for Maple and Mathematica
- ...

Standalone applications:

- SHEEP, Classi, STensor, based on Lisp, developed in 1970's and 1980's (free) [R. d'Inverno, I. Frick, J. Åman, J. Skea, et al.]
- Cadabra field theory (free) [K. Peeters]
- SnapPy topology and geometry of 3-manifolds, based on Python (free) [M. Culler, N. M. Dunfield & J. R. Weeks]
- ...

cf. the complete list at http://www.xact.es/links.html

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The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of SageMath

SageMath is free

Freedom means

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- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful *object oriented language*, with a neat syntax

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SageMath is developing and spreading fast

...sustained by an enthousiast community of developers

Object-oriented notation in Python

As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

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result = object.function(arguments)
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In a procedural language, this would be written as

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Examples

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1

SageMath approach to computer mathematics

SageMath relies on a `Parent` / `Element` scheme: each object x on which some calculus is performed has a "parent", which is another SageMath object X representing the set to which x belongs.
The calculus rules on x are determined by the *algebraic structure* of X.
Conversion rules prior to an operation, e.g. $x + y$ with x and y having different parents, are defined at the level of the parents

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Example

```
sage: x = 4 ; x.parent()
Integer Ring
sage: y = 4/3 ; y.parent()
Rational Field
sage: s = x + y ; s.parent()
Rational Field
sage: y.parent().has_coerce_map_from(x.parent())
True
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This approach is similar to that of Magma and is different from that of Mathematica, in which everything is a tree of symbols

Outline

The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

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Concretely, the project amounts to creating new Python classes, such as `Manifold`, `Chart`, `TensorField` or `Metric`, within SageMath's **Parent/Element framework**.

Implementing coordinate charts

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Examples:

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- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$

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In SageManifolds, an arbitrary number of charts can be introduced

To fully specify the manifold, one shall also provide the *transition maps* on overlapping chart domains (SageManifolds class `CoordChange`)

Implementing scalar fields

A **scalar field** on manifold M is a smooth mapping

$$\begin{array}{rccc} f: & U \subset M & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & f(p) \end{array}$$

where U is an open subset of M.

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The various coordinate representations F, $\hat{F}$, ... of f are stored as a *Python dictionary* whose keys are the charts C, $\hat{C}$, ...:

$$f.\_\mathtt{express} = \left\{ C : F, \ \hat{C} : \hat{F}, \ldots \right\}$$

with $\underbrace{f(\,p\,)}_{\text{point}} = F(\underbrace{x^1, \ldots, x^n}_{\text{coord. of } p \text{ in chart } C}) = \hat{F}(\underbrace{\hat{x}^1, \ldots, \hat{x}^n}_{\text{coord. of } p \text{ in chart } \hat{C}}) = \ldots$

The scalar field algebra

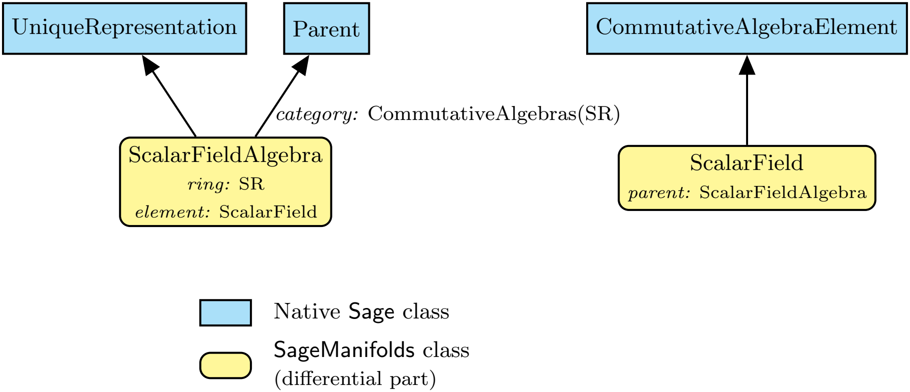
Given an open subset $U \subset M$, the set $C^\infty(U)$ of scalar fields defined on U has naturally the structure of a **commutative algebra over** $\mathbb{R}$:

1. it is clearly a vector space over $\mathbb{R}$
2. it is endowed with a commutative ring structure by pointwise multiplication:

$$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f.g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class `ScalarFieldAlgebra`.

Classes for scalar fields



Outline

Various ways to install/access SageMath

- Install on your computer:
  3 options:
  - compile from source (Linux, MacOS X):
    ```
    git clone git://github.com/sagemath/sage.git
    cd sage
    MAKE='make -j8' make
    ```
  - install a compiled binary version (Linux, MacOS X)
  - run in virtual machine (Windows)
- Sage Debian Live USB key:
  http://sagedebianlive.metelu.net/
  comes along with SageMath (boosted with octave, scilab), Geogebra, LaTex, gimp, vlc, LibreOffice,...
- SageMathCloud:
  https://cloud.sagemath.com/
- SageMathCell:
  Single cell mode: http://sagecell.sagemath.org/

Various ways to run SageMath

- Console mode:
  run the command `sage`
- Standard Sage Notebook:
  run the command `sage -n`
  $\Longrightarrow$ worksheet file format: `sws`
- Jupyter Notebook[1]:
  run the command `sage -n jupyter`
  $\Longrightarrow$ worksheet file format: `ipynb`
- SageMathCloud:
  in your browser, open `https://cloud.sagemath.com/`
  $\Longrightarrow$ worksheet file format: `sagews`, `ipynb`

[1] the future standard notebook

Deriving and solving the TOV equations

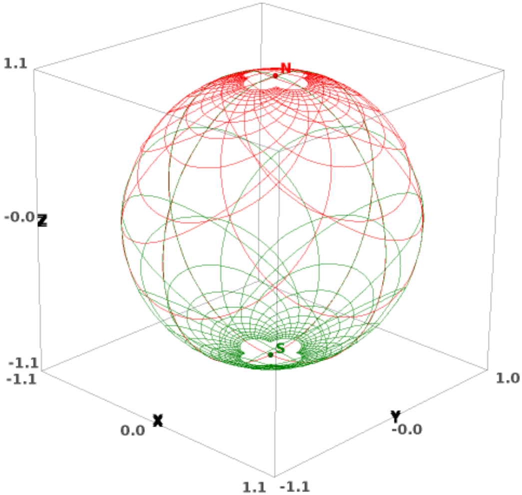
See the worksheet at
http://nbviewer.jupyter.org/github/egourgoulhon/NewCompStarSchool/blob/master/WorkSheets/TOV.ipynb

The source is stored at GitHub, from which it can be downloaded:
https://github.com/egourgoulhon/NewCompStarSchool

A copy of the worksheet is also publicly available on the SageMathCloud (click on the icon “Files”):
https://cloud.sagemath.com/projects/8f20b8d0-aac0-4454-95d5-dc929acae1e5/files/TOV.ipynb

Outline

The 2-sphere example



Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

← picture obtained via function `Chart.plot()`

See the worksheet at
http://sagemanifolds.obspm.fr/examples/html/SM_sphere_S2.html

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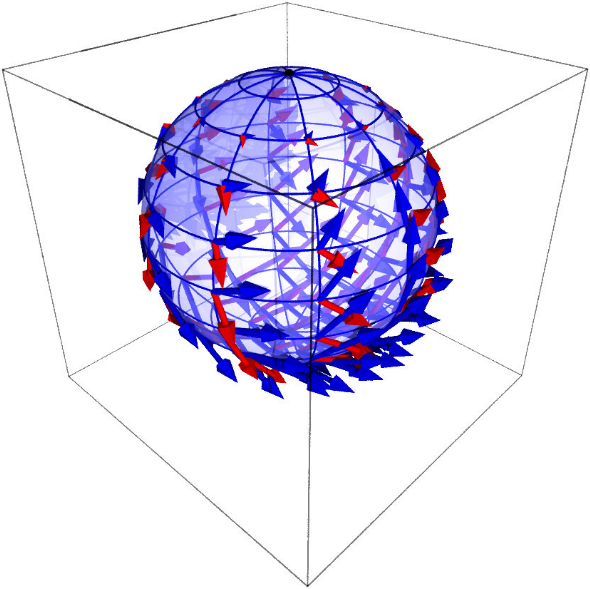
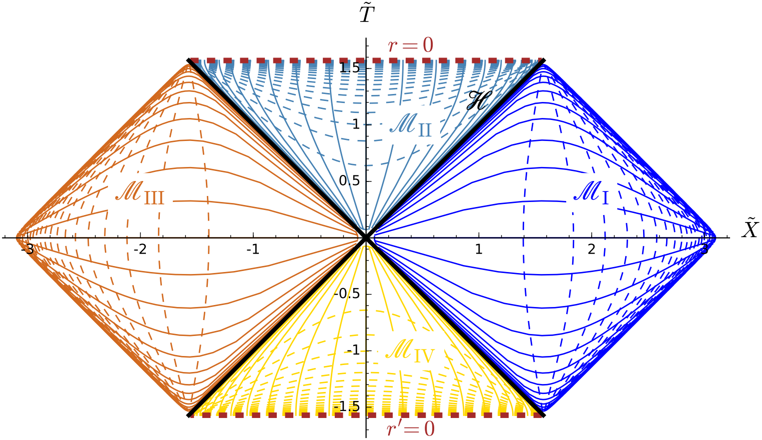


Vector frame associated with the stereographic coordinates (x, y) from the North pole

- $\frac{\partial}{\partial x}$
- $\frac{\partial}{\partial y}$

← picture obtained via function `VectorField.plot()`

Charts on Schwarzschild spacetime
The Carter-Penrose diagram



Two charts of standard Schwarzschild-Droste coordinates (t, r, θ, φ) plotted in terms of compactified coordinates $(\tilde{T}, \tilde{X}, \theta, \varphi)$; see the worksheet at
http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html

Outline

Conclusion and perspectives

- **SageManifolds** is a work in progress
  $\sim$ 64,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.9) is freely available (GPL) at
  http://sagemanifolds.obspm.fr/

Current status

Already present (v0.9):

- maps between manifolds, pullback operator
- submanifolds, pushforward operator
- curves in manifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- all monoterm tensor symmetries
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections, curvature, torsion
- pseudo-Riemannian metrics, Weyl tensor
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

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- *Not implemented yet (but should be soon):*
  - extrinsic geometry of pseudo-Riemannian submanifolds
  - computation of geodesics (numerical integration via SageMath/GSL or `Gyoto`)
  - integrals on submanifolds

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- *Future prospects:*
  - add more graphical outputs
  - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
  - connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

Integration into SageMath

SageManifolds is aimed to be fully integrated into SageMath

- The algebraic part (tensors on free modules of finite rank) has been submitted to SageMath Trac as ticket #15916 and got a positive review $\Longrightarrow$ integrated in SageMath 6.6
- The differential part has been split in various tickets for submission to SageMath Trac (cf. the metaticket #18528); 4 tickets have been already accepted and integrated in SageMath 7.3
- Until complete integration, the full SageManifold has to be downloaded from http://sagemanifolds.obspm.fr/
- SageManifolds v0.9 is installed in the SageMathCloud $\Longrightarrow$ open a free account and use it online: https://cloud.sagemath.com/

Want to join the project or simply to stay tuned?

visit http://sagemanifolds.obspm.fr/
(download page, documentation, example worksheets, mailing list)