Relativistic perfect fluids

- Kinematics

First fundamental quantity is fluid's 4-velocity

$u^\mu := \frac{dx^\mu}{d\tau}$ τ: fluid proper time, ie time in fluid's reference frame

$a^\mu := u^\nu \nabla_\nu u^\mu$: fluid's 4-acceleration : (variation of u along fluid's worldline)

$\underline{u} \cdot \underline{u} = -1$: timelike unit vector

proof $u^\mu u_\mu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = -1$

$\underline{a}$ and $\underline{u}$ are orthogonal, ie $\underline{a} \cdot \underline{u} = 0$

proof : first note that

$$\nabla_\mu (u^\nu u_\nu) = \nabla_\mu (-1) = 0 = (\nabla_\mu u^\nu) u_\nu + (\nabla_\mu u_\nu) u^\nu = 2 (\nabla_\mu u^\nu) u_\nu \Rightarrow$$

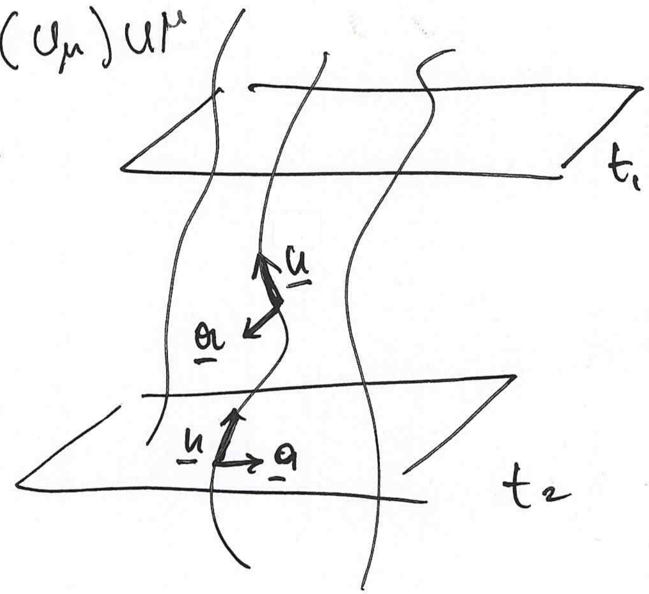
$$u^\nu \nabla_\mu u_\nu = 0 \; ;$$

Then

$$a^\mu u_\mu = (u^\nu \nabla_\nu u^\mu) u_\mu = u^\nu \nabla_\nu (u_\mu u^\mu) - u^\nu \nabla_\nu (u_\mu) u^\mu$$

$$= 0 - u^\nu \nabla_\nu (u^\mu) u_\mu$$

$$= 0 \qquad \text{qed.}$$

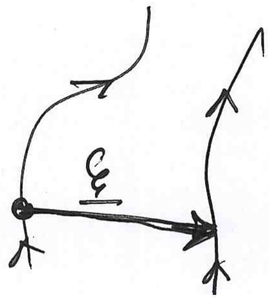


Without loss of generality we can write

$$u^\alpha = u^0 \left(1, \frac{dx^i}{dt}\right) = u^0 (1, v^i)$$

All kinematic properties of the fluid can be expressed in terms of $\underline{u}$ and $\underline{a}$

Let $\underline{\xi}$ be the 4-vector between two neighbouring worldlines



<u>displacement vector</u>

Lie derivative. Does anyone remember?

The Lie derivative expresses the covariant derivative of a vector (tensor) field relative to another vector (tensor) field

$$\mathcal{L}_{\underline{V}}\,\underline{U} = \underline{V}\,\partial\,\underline{U} - \underline{U}\,\partial\,\underline{V} = -[\underline{U}, \underline{V}] = [\underline{V}, \underline{U}]$$

In component form

$$(\mathcal{L}_{\underline{V}}\underline{U})^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu U^\mu - U^\nu \nabla_\nu V^\mu$$

$$(\mathcal{L}_{\underline{V}}\underline{U})_\mu = V^\nu \partial_\nu U_\mu + U^\nu \partial_\nu V_\mu = V^\nu \nabla_\nu U_\mu + U^\nu \nabla_\nu V_\mu$$

Properties

- $\mathcal{L}_{\phi \underline{V}} \underline{T} = \phi \, \mathcal{L}_{\underline{V}} \underline{T}$

- $\mathcal{L}_{\underline{V}} \phi = V^\nu \partial_\nu \phi$

- $\mathcal{L}_{\underline{V}} (a Y^{\alpha\gamma} + b Z^{\beta\gamma}) = a \mathcal{L}_{\underline{V}} Y^{\alpha\gamma} + b \mathcal{L}_{\underline{V}} Z^{\beta\gamma}$

- $\mathcal{L}_{\underline{V}} (Z^{\mu\nu} Y_{\alpha\beta}) = \mathcal{L}_{\underline{V}} (Z^{\mu\nu}) Y_{\alpha\beta} + Z^{\mu\nu} \mathcal{L}_{\underline{V}} Y_{\alpha\beta}$

- $\mathcal{L}_V T^{\alpha}{}_{\beta} = V^{\mu} \partial_{\mu} T^{\alpha}{}_{\beta} - T_{\beta}{}^{\mu} \partial_{\mu} X^{\alpha} + T^{\alpha}{}_{\mu} \partial_{\beta} V^{\mu}$

$\underline{\xi}$ is Lie dragged along $\underline{u}$ $\iff$

$$\mathcal{L}_{\underline{u}}\,\underline{\xi} = 0 \iff \mathcal{L}_{\underline{u}}\,\underline{\xi} = [\underline{u}, \underline{\xi}] = 0 \iff$$ (Exercise)

$$(\mathcal{L}_{\underline{u}}\,\underline{\xi})^{\nu} = u^{\mu}\nabla_{\mu}\xi^{\nu} - \xi^{\mu}\nabla_{\mu}u^{\nu} = 0 \implies$$

$$\dot{\xi}^{\mu} := u^{\nu}\nabla_{\nu}\xi^{\mu} = \xi^{\nu}\nabla_{\nu}u^{\mu} \implies$$

convective derivative of $\underline{u}$ along $\underline{\xi}$ = convective deriv. of $\underline{\xi}$ along $\underline{u}$

$$\boxed{\nabla_{\nu}u_{\mu} = \left(\omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - a_{\mu}u_{\nu}\right)}$$

irreducible decomposition of generic rank-2 tensor $[\nabla_{\nu}u_{\mu}]$

In other words, we have shown that the convective derivative of the displacement vector can be decomposed in terms of three fundamental tensors $\underline{\omega}$, $\underline{\sigma}$, Θ.

$$\omega_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu \nabla_{[\beta} u_{\alpha]} = h^\alpha{}_\mu h^\beta{}_\nu \frac{1}{2}\left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right)$$

where $\underline{h}$ is the projector orthogonal to $\underline{u}$

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu \quad ; \quad \underline{h}\cdot\underline{u} = 0 = h_{\mu\nu} u^\mu = g_{\mu\nu} u^\mu + u_\mu u_\nu u^\mu$$
$$= u_\nu - u_\nu = 0.$$

$\boxed{\underline{\omega}: \text{ kinematic vorticity tensor}}$

$$\sigma_{\mu\nu} := \nabla_{\langle\mu} u_{\nu\rangle} = \nabla_{(\mu} u_{\nu)} + a_{(\mu} u_{\nu)} - \frac{1}{3}\Theta h_{\mu\nu}$$
$$= \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu + a_\mu u_\nu + a_\nu u_\mu - \frac{2}{3}\Theta h_{\mu\nu}\right)$$

$\boxed{\underline{\sigma}: \text{ shear tensor}}$

$$\Theta := h^{\mu\nu} \nabla_\nu u_\nu = \nabla_\mu u^\mu$$

$\boxed{\Theta: \text{ expansion scalar}}$

$\underline{\omega}$, $\underline{\sigma}$ are anti-symmetric and symmetric tensors respectively, and satisfy a number of identities (EXERCISE)

If a fluid element is thought of as an ellipsoid, then the vorticity tensor represents rigid rotations of the principal axis wrt the inertial frame

In the frame determined by $\underline{u}$, the dual of $\underline{\omega}$ ie

$$ {}^{*}\omega^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \omega_{\alpha\beta} \qquad \epsilon^{\mu\nu\alpha\beta} : \text{Levi-Civita tensor}^{(1)} $$

$$ \omega^{\mu} := {}^{*}\omega^{\mu\nu} u_{\nu} \quad : \text{vorticity 4-vector} $$

$$ \stackrel{!}{=} \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \omega_{\alpha\beta} u_{\nu} $$

vorticity 4-vector ; equivalent of classical vorticity 3-vector

(1)

$$ \epsilon_{\alpha\beta\mu\nu} = -\sqrt{-g}\, \eta_{\alpha\beta\mu\nu} $$

$$ \epsilon^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{-g}} \eta^{\alpha\beta\mu\nu} $$

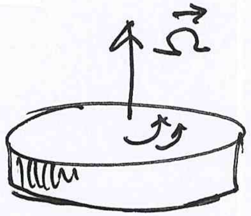
$$ \eta_{\alpha\beta\mu\nu} = \begin{cases} 1 & \text{even permutation of } 0123 \\ -1 & \text{odd permutation of } 0123 \\ 0 & \text{if } \alpha\beta\mu\nu \text{ not all diff.} \end{cases} $$

$\vec{\omega} = \vec{\nabla} \times \vec{v}$

$\vec{\omega} = (0, 0, \omega^z) = (0, 0, 2\Omega)$

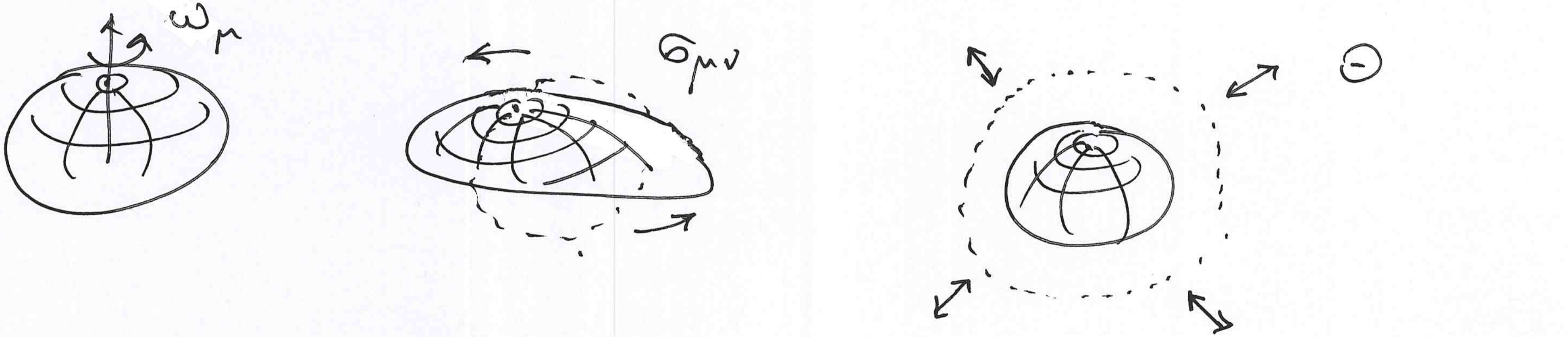
cylind. coord sys

Eg



The shear tensor measures the changes in the changes in the ellipsoid's axes while preserving the volume. (change in shape but not in volume).

Conversely, the expansion tensor measures changes in volume without changing shape



Energy momentum tensor

we have seen that if the distribution function is known we can define the 1st, and 2nd moment of the distribution as

$$J^\mu = m N^\mu = mc \int p^\mu f \frac{d^3p}{p^0}$$: rest-mass density current (1st moment)

$$T^{\mu\nu} = c \int p^\mu p^\nu f \frac{d^3p}{p^0}$$: energy-momentum tensor (2nd moment)

However we can also obtain a different definition of these tensors without having to consider a distribution function. This is possible if we remember that in a rest-frame comoving with the fluid (indicated with hatted indices)

$J^{\hat{\mu}}$: flux of rest-mass density current in $\hat{\mu}$-direction

$T^{\hat{\mu}\hat{\nu}}$: flux of $\hat{\mu}$ momentum density in the $\hat{\nu}$-direction

In this way we have $J^{\hat{\mu}} = (\rho, 0, 0, 0)$

$T^{\hat{0}\hat{0}}$: (total) energy density

$T^{\hat{0}\hat{i}}$: flux of energy density in $\hat{i}$-th direction

$T^{\hat{i}\hat{0}}$: flux of $\hat{i}$-th mom. density in $\hat{0}$-th direction

$\Big\}\ T^{\hat{0}\hat{i}} = T^{\hat{i}\hat{0}}$

$T^{\hat{i}\hat{j}}$: flux of $\hat{j}$-th " density in $\hat{i}$-th direction

More specifically if $E = \langle p^{\hat{0}} \rangle$: energy of particles in fluid element, then

$$T^{\hat{0}\hat{0}} = n\langle p^{\hat{0}}\rangle = e$$

$$T^{\hat{0}\hat{i}} = 0 \qquad \text{(perfect fluid)}$$

$$T^{\hat{i}\hat{0}} = 0\ (i \neq j) \qquad \text{(symmetric tensor)}$$

$$T^{\hat{i}\hat{j}} = p\ (i = j) \qquad \text{isotropic pressure}$$

$$T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

More generally

$$T^{\mu\nu} = \begin{pmatrix} \boxed{T^{00}} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & \cdots \\ T^{20} & \cdots & T^{22} & \cdots \\ T^{30} & \cdots & \cdots & T^{33} \end{pmatrix}$$

energy flux

momentum flux

isotropic pressure

momentum density

without loss of generality we can express the above definitions using the fluid 4-velocity $\underline{u}$; in the comoving frame $u^{\hat{\mu}} = (1,0,0,0)$

in the fluid rest-frame metric is locally flat

$$(\cdot)\quad \begin{cases} J^{\hat{\mu}} = \rho u^{\hat{\mu}} \\ T^{\hat{\mu}\hat{\nu}} = e u^{\hat{\mu}} u^{\hat{\nu}} + p(\eta^{\hat{\mu}\hat{\nu}} + u^{\hat{\mu}} u^{\hat{\nu}}) \end{cases}$$

Eg $T^{\hat{i}\hat{j}} = e u^{\hat{i}} u^{\hat{j}} + p(\eta^{\hat{i}\hat{j}} + u^{\hat{i}} u^{\hat{j}}) = p\,\eta^{\hat{i}\hat{j}} = \begin{cases} p & \hat{i}=\hat{j} \\ 0 & \hat{i}\neq\hat{j} \end{cases}$

Because of covariance we can generalize $(\cdot)$ to any frame
ie $\eta^{\mu\nu} \to g^{\mu\nu}$; $u^{\hat{\mu}} \to u^{\mu}$

$$(\cdot\cdot)\quad \begin{cases} J^{\mu} = \rho u^{\mu} \\ T^{\mu\nu} = e u^{\mu} u^{\nu} + p(g^{\mu\nu} + u^{\mu} u^{\nu}) \end{cases}$$

$$= (e+p) u^{\mu} u^{\nu} + p g^{\mu\nu}$$

The definitions (..) can also be interpreted in the light of the considerations made above for the energy-mom. tensor. To this scope we need to decompose $\underline{T}$ in the direction defined by $\underline{u}$ and in the one orthogonal to it. To this scope we can introduce the <u>projection tensor</u> $\underline{h}$

$$h_{\mu\nu} := u_\mu u_\nu + g_{\mu\nu} \quad \Longrightarrow \quad h^\mu{}_\nu = u^\mu u_\nu + \delta^\mu{}_\nu$$

where $\quad \underline{h}\cdot\underline{u} = 0 = (u_\mu u_\nu + g_{\mu\nu})\, u^\mu = -u_\nu + u_\nu$

As a result we can introduce

$\underline{h}\cdot\underline{T}\cdot\underline{h}$

$\mathcal{L}_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}$: fully spatial stress tensor $\quad (\mathcal{L}_{\mu\nu} u^\nu = 0)$

$\mathcal{L}_\mu := -h^\alpha{}_\mu u^\beta T_{\alpha\beta}$: spatial momentum density

$L := \mathcal{L}^\mu{}_\mu$

$e := u^\alpha u^\beta T_{\alpha\beta}$: energy density (projection of $\underline{T}$ in $\underline{u}$ direction)

With these definitions it is possible to construct a ~~general~~ (geometric) expression for the energy-momentum tensor

$$T_{\mu\nu} = e\, u_\mu u_\nu + 2\, u_{(\mu} L_{\nu)} + L_{\mu\nu}$$

Specializing to a perfect fluid: $L_\nu = 0$ and $L_{\mu\nu} = p h_{\mu\nu}$ $\square$

Having now an explicit expression for the energy density and pressure in terms of the energy-momentum density, we can go back and obtain these quantities directly from the distribution function

$$e = -u_\mu J^\mu = -m\, u_\mu \int p^\mu f \frac{d^3p}{p^0}$$

$$e = u_\mu u_\nu T^{\mu\nu} = u_\mu u_\nu \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

$$p = \frac{1}{3}\left(\eta_{\mu\nu} + u_\mu u_\nu\right) T^{\mu\nu} = \frac{1}{3}\left(\eta_{\mu\nu} + u_\mu u_\nu\right) \int p^\mu p^\nu f \frac{d^3p}{p^0}$$

Relativistic hydrodynamics equations

$$\nabla_\mu J^\mu = 0 = \nabla_\mu (\rho u^\mu) \iff u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0 \quad : \ 1 \text{ eq.}$$

$$\nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu \left[(e+p) u_\mu u_\nu + p g_{\mu\nu} \right] = 0$$

$$= \nabla_\mu \left(\rho h u_\mu u_\nu + p g_{\mu\nu} \right) = \underline{\nabla} \cdot \underline{T} = 0$$

4 eqs

5 eqs in 6 unknowns

$$\underbrace{(\underset{3}{u^\mu}, \underset{1}{\rho}, \underset{1}{p}, \underset{1}{\epsilon})}$$

one EoS needed to close the system.

Conservation of energy and momentum are obtained after projecting $\underline{\nabla} \cdot \underline{T}$ in the direction along and orthogonal to $\underline{u}$

$$\underline{h} \cdot \underline{\nabla} \cdot \underline{T} = 0 \iff$$

$$h^\nu{}_\lambda \nabla_\mu T^{\mu\lambda} = h^\nu{}_\lambda \left[u^\lambda u^\mu \nabla_\mu (\rho h) + \rho h u^\mu \nabla_\mu u^\lambda + \rho h \Theta u^\lambda + g^{\lambda\mu} \nabla_\mu p \right]$$

$$= \rho h u^\mu \nabla_\mu u^\nu + h^\nu{}_\lambda g^{\lambda\mu} \nabla_\mu p = 0$$

$\underline{h} \cdot \underline{u} = 0$

$\Rightarrow$

(Exercise: derive Newtonian limit of eqs (Δ) and ($*$))

$$(\Delta) \quad \boxed{u^\mu \nabla_\mu u_\nu + \frac{h^\mu{}_\nu}{\rho h} \nabla_\mu p = 0} \iff$$

$$\boxed{\rho h a^\mu = -(g^{\mu\nu} + u^\mu u^\nu) \nabla_\nu p}$$

relativistic Euler equation ☉

$a^\mu = 0$ if $p = 0$ or $\nabla_\mu p = 0$

i.e. geodesic motion for uniform pressure or dust.

Similarly

$$\underline{u} \cdot \underline{\nabla} \cdot \underline{T} = 0 \iff$$

$$u^\mu \nabla_\mu e + \rho h \Theta = 0$$

Recalling that the continuity equation implies $\Theta = -\frac{1}{\rho} u^\mu \nabla_\mu \rho$, so that

$$(*) \quad \boxed{u^\mu \nabla_\mu e - h u^\mu \nabla_\mu \rho = 0}$$

relativistic energy conservation eq. ⊙⊙

☉ $\partial_t v^j + v^i \partial_i v^j = -\frac{1}{\rho} \partial_j p$

⊙⊙ $\partial_t \left(\frac{1}{2}\rho v^2 + \rho \epsilon\right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right)\vec{v}\right] = 0$

Let's recall the 1st law of thermodynamics

$$de = h\,d\rho + \rho T ds$$

Then it's clear that (*) implies $\boxed{u^\mu \nabla_\mu s = 0}$ specific entropic is conserved along fluid lines

Stated differently: <u>perfect fluids are adiabatic</u>

($\nabla_\mu s = 0 \iff$ isentropic fluid).

Perfect fluids and symmetries.

Theorem

Let $\underline{T}$ satisfy a conservation equation $\underline{\nabla}\cdot\underline{T}=0$ and $\underline{\xi}$ be a Killing vector field $\odot$, then the quantity $\underline{Q}:=\underline{T}\cdot\underline{\xi}$ also satisfies a conservation law, ie $\underline{\nabla}\cdot\underline{Q}=0$

Proof.

$$\underline{\nabla}\cdot\underline{Q} = \nabla_\mu Q^\mu = \nabla_\mu(T^{\mu\nu}\xi_\nu) = (\nabla_\mu T^{\mu\nu})\xi_\nu + T^{\mu\nu}\nabla_\mu \xi_\nu$$

$$= 0 - T^{\mu\nu}\nabla_\nu \xi_\mu$$

Killing vector field

A vector field $\underline{\xi}$ is said to be a Killing field if

$\mathcal{L}_{\underline{\xi}}\, \underline{g} = 0$: the metric is Lie dragged along $\underline{\xi}$

$$\mathcal{L}_{\underline{\xi}}\, \underline{g} = \xi^{\alpha} \partial_{\alpha} g_{\mu\nu} + g_{\mu\alpha} \partial_{\nu} \xi^{\alpha} + g_{\alpha\nu} \partial_{\mu} \xi^{\alpha} = 0$$

$\underline{\xi}$: generator of the associated symmetry group G

$\mathcal{L}_{\underline{\xi}}\, \underline{g} = 0 \iff \nabla_{(\mu} \xi_{\nu)} = 0$: Killing equations (define isometries of the spacetime).

Ex.

$\xi^{\mu} = (1,0,0,0)$: time like Killing vector. Then

$\partial_t g_{\mu\nu} = 0$: the metric is time independent

To derive the general result of perfect fluids in the presence of symmetries let's go back to the momentum conservation equation

$$u^\mu \nabla_\mu u_\nu + \frac{1}{\rho h} h^\mu{}_\nu \nabla_\mu p = 0$$

which can be written as

$$\rho u^\mu \nabla_\mu (h u_\nu) - \rho u_\nu u^\mu \nabla_\mu h = - \nabla_\nu p - u^\mu u_\nu \nabla_\mu p$$

A perfect fluid is adiabatic, ie $u^\mu \nabla_\mu s = 0 \Rightarrow dp = \rho dh \Longleftrightarrow$

$\qquad u^\mu u_\nu \nabla_\mu p = u^\mu u_\nu \rho dh$

$\qquad$ (2nd - 4th term cancel)

$$\Rightarrow \quad u^\mu \nabla_\mu (h u_\nu) = - \frac{\nabla_\nu p}{\rho} \qquad (*)$$

Recall now that

$$\mathcal{L}_{\underline{u}} u_\nu = u^\mu \nabla_\mu u_\nu + \underbrace{u_\mu \nabla_\nu u^\mu}_{=0\ \boxed{\cdot}} = u^\mu \nabla_\mu u_\nu = a_\nu$$

$\boxed{\cdot}$ $u_\mu \nabla_\nu u^\mu = \nabla_\nu (u_\mu u^\mu) - u^\mu \nabla_\nu u_\mu = - u^\mu \nabla_\nu u_\mu \Rightarrow 2 u^\mu \nabla_\nu u^\mu \Rightarrow$ qed

As a result (*) can be written as

$$\mathcal{L}_{\underline{u}} h$$

$$\mathcal{L}_{\underline{u}} (h u_\mu) = - \frac{1}{\varrho} \nabla_\mu p - \nabla_\mu h$$

Contracting with a killing field

$$\xi^\mu \mathcal{L}_{\underline{u}} (h u_\mu) = \mathcal{L}_{\underline{u}} (\xi^\mu h u_\mu) - h u_\mu \mathcal{L}_{\underline{u}} \xi^\mu$$

$$= \mathcal{L}_{\underline{u}} (h \xi^\mu u_\mu) - h u^\mu u^\nu \nabla_{(\nu} \xi_{\mu)}$$

$$= \mathcal{L}_{\underline{u}} (h \xi^\mu u_\mu)$$

As a result

$$\mathcal{L}_{\underline{u}} (h \xi^\mu u_\mu) = - \frac{\xi^\mu \nabla_\mu p}{\varrho} - \xi^\mu \nabla_\mu h = - \frac{1}{\varrho} \mathcal{L}_{\underline{\xi}} p - \mathcal{L}_{\underline{\xi}} h$$

If the fluid satisfies the same symmetries of the spacetime, the $\mathcal{L}_{\underline{\xi}} \underline{B} = 0$ $\quad \underline{B}$ generic fluid-related quantity

$$\Rightarrow \mathcal{L}_{\underline{\xi}}\, p = 0 = \mathcal{L}_{\underline{\xi}}\, h \quad \Rightarrow$$

$$\boxed{\mathcal{L}_{\underline{u}}\left(h\, \xi^{\mu} u_{\mu}\right) = 0 = \nabla_{\underline{u}}\left(h\, \xi^{\mu} u_{\mu}\right) = 0} \qquad (**)$$

In other words, if $\underline{\xi}$ is a symmetry generator of the spacetime and the fluid shares the same symmetry, then the scalar quantity $h\, \underline{u} \cdot \underline{\xi}$ is conserved along the fluid lines.

Expression (**) is similar to the condition for geodesic curves with tangent $\underline{u}$; in that case in fact

$$\mathcal{L}_{\underline{u}}\left(u_{\mu} \xi^{\mu}\right) = 0 \qquad \underline{u} \cdot \underline{\xi} \text{ is conserved along the geodesic.}$$

(**) is the fluid extension of the conservation of some quantity

Ex. $\underline{\xi} = \frac{\partial}{\partial \phi}$, ie $\xi^{\mu} = (0, 0, 0, 1)$ in spherical coords.

Then $h\, u_\mu\, \xi^\mu = h\, u_\phi$ is conserved

$\hookrightarrow$ specific angular momentum

$$\underset{\text{Newtonian limit}}{h\, u_\phi \longrightarrow} \Omega r^2 = \text{conserved}, \quad \text{where } \Omega = d\phi/dt$$

All of this tensor algebra has not been in vein because the relativistic Bernoulli theorem is now trivial to derive.

For a stationary flow, there exist a timelike killing vector $\underline{\xi} = \partial_t$; $\xi^\mu = (1,0,0,0) \Rightarrow$

Bernoulli's theorem $\boxed{\mathcal{L}_{\underline{u}}\, (h\, \underline{u} \cdot \underline{\xi}) = \mathcal{L}_{\underline{u}}\, (h\, u_t) = 0}$ $\boxed{h u_t}$ is the Bernoulli constant and is conserved along fluid lines

Relativistic hydrodynamics

We have seen the corresponding eqs are

$$(*) \quad \nabla_\mu (J^\mu) = \nabla_\mu (\rho u^\mu) = 0$$

$$(**) \quad \nabla_\mu T^{\mu\nu} = 0 \quad \begin{cases} h^\alpha{}_\mu \nabla_\nu T^{\mu\nu} = 0 & \text{mom. conservation} \\ \\ u_\mu \nabla_\nu T^{\mu\nu} = 0 & \text{energy conservation} \end{cases}$$

These equations are nonlinear and to better appreciate the implications of this statement, let me recall that (*), (**) can be written in a general form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S}$$

A large class of equations in mathematical physics (eg. EFEs, eqs. of hydrodynamics and MHD) can be written in a compact form as

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = \underline{S} \qquad (*)$$

$$\partial_t U_J + (A^i)_{JK} \nabla_i U_K = S_J$$

where $\underline{U} = \{U_1, U_2, \dots U_J\}$: state vector

$\underline{S} = \{S_1, S_2, \dots S_J\}$: source term

A : matrix of coefficients and

The properties of the system (*) depend on the properties of A, and S.

(i) $\left\{ \begin{array}{l} a_{JK} \ : \text{elements of } A \\ a_{JK} = \text{const.} \ ; \quad S_J = \text{const} \end{array} \right\}$ (*) is a LINEAR system of equations with constant coefficients

(ii) $\left\{ a_{JK} = a_{JK}(x,t) \ ; \quad S_J = S_J(x,t) \right\}$ (*) is a LINEAR system with variable coefficients

(iii) $A = A(\underline{u})$ (*) is a non LINEAR system (or quasi-linear)

(often referred to as quasi-linear)

More importantly, the system (*) is said to be (strongly) HYPERBOLIC if A is diagonalizable with a set of real eigenvalues $\lambda_1, \ldots \lambda_N$ and a set of N linearly independent right eigenvectors, ie if

$$\Lambda := R^{-1} A R = \text{diag}(\lambda_1, \lambda_2 \ldots \lambda_N)$$

R: matrix of right eigenvectors $R^{(i)}$

$$A R^{(i)} = \lambda_i R^{(i)}$$

$\lambda_i \in \mathbb{R}$: real eigenvalues

(*) is said to be <u>STRICTLY HYPERBOLIC</u> if $\lambda_{(i)}$ are real and distinct

(*) is said to be <u>SYMMETRIC HYPERBOLIC</u> if A is symmetric, ie $A = A^T$

(*) is said to be <u>WEAKLY HYPERBOLIC</u> if A is not diagonalizable

Examples of hyperbolic equations are

- advection equation $\partial_t u + v\,\partial_x u = 0$
- wave equation $\partial_t^2 u - v^2 \partial_x^2 u = 0$
- hydrodynamic equations (inviscid)
- Einstein equations (only in harmonic coordinates $\Box x^\alpha = 0$)

The importance of hyperbolicity is strictly related with that of WELL POSEDENESS of the Cauchy initial - value problem.

$\underline{u}(x,0)$: initial data

$\underline{u}(x,t)$: solution of set (*) at time t

(*) is well posed if

$$\| u(x,t) \| \leq k e^{at} \| u(x,0) \|$$

$k, a \in \mathbb{R}$: constants.

In other words the solution is always bounded by some exponential of the initial data ("it does not blow up...!")

An important theorem of hyperbolic systems states

$$\boxed{\begin{array}{l}(*) \text{ a hyperbolic set} \\ \quad \text{of equations}\end{array} \implies (*) \text{ is well posed}}$$

Opposite implication not true.

It follows that a weakly hyperbolic system is not guaranteed to be well-posed and indeed the numerical solution leads to the growth of unstable modes ("codes crash"...)

Let's go back to Newtonian hydrodynamics

$$\begin{cases} \partial_t \rho + v^i \partial_i \rho + \rho \, \partial_i v^i = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{1}{\rho} \partial^i p = 0 \\ \partial_t s + v^i \partial_i s = 0 \end{cases} \iff \begin{cases} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho} \vec{\nabla} p = 0 \\ \partial_t s + (\vec{v} \cdot \vec{\nabla}) s = 0 \end{cases}$$

s: specif entropy $= \frac{S}{m}$

This set can be written in the general form (see previou lecture)

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0 \qquad (*)$$

A: 3
5×5
matrices

where $\underline{u} = \{\rho\, v^i, s\}^T = \begin{pmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ s \end{pmatrix}$: state vector

$$A^1 = \begin{pmatrix} v' & \rho & 0 & 0 & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v' & 0 & 0 & \frac{1}{\rho}\partial_s p \\ & & v' & & \\ & 0 & & v' & 0 \\ & & & & v' \end{pmatrix}$$

$$A^2 = \dots$$

$$A^3 = \dots$$

We have already discussed that the system (*) is <u>nonlinear</u> if the coefficients of A are functions of the state vector $a_{jk} = a_{jk}(\underline{u})$ while the system is <u>linear</u> if the coefficients are constant.

There is no better way to appreciate the difference between linear and nonlinear hyperbolic equations than to consider some examples.

Let's start with a <u>linear</u> hyperbolic equation: advection equation in 1+1 spacetime

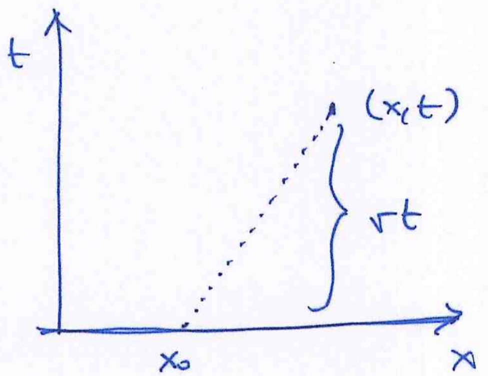
$$\boxed{\partial_t u + v\,\partial_x u = 0} \qquad v = \text{const.}$$

A has only one component $a_{11} = v = \text{const}$ (linear!)

This equation has the simple solution

$$u(x,t) = u(x_0, 0) = u_0(x_0) = u_0(x - vt)$$

$$x_0 = x - vt$$



$$v = \frac{dx}{dt}$$

In other words the solution at any new time and position $u(x,t)$ can be computed from the initial solution u_0 at the position x_0 suitably translated in space-time.

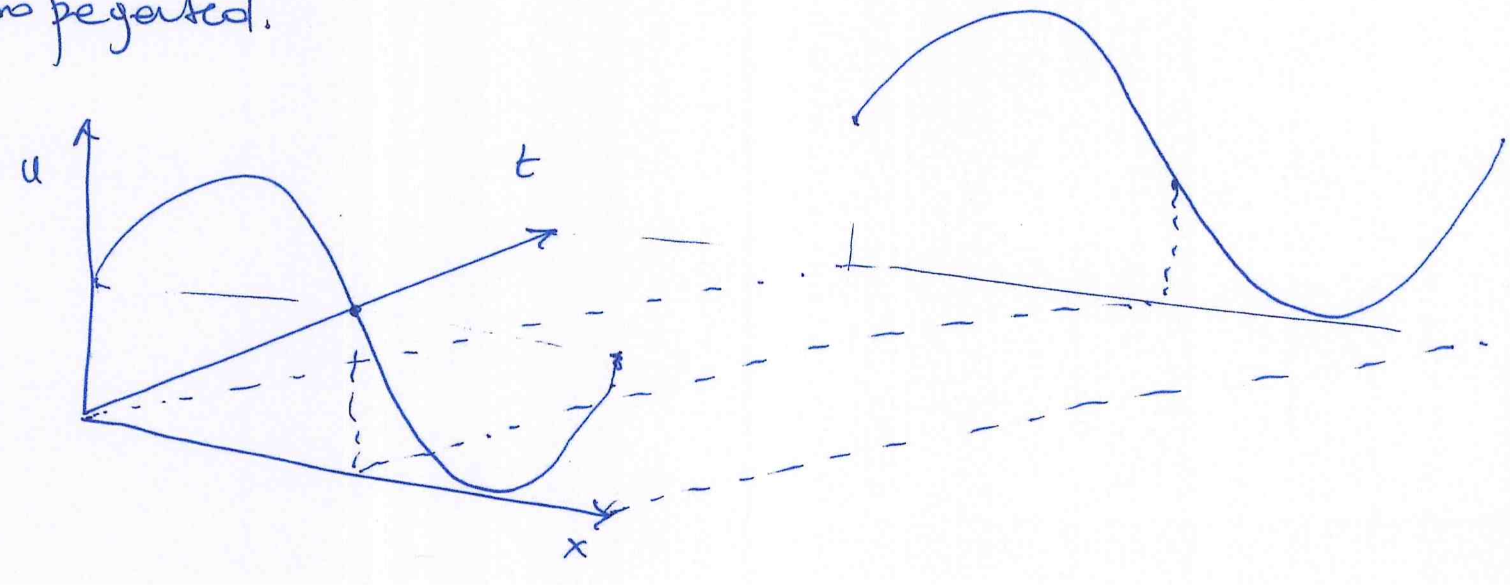
We can actually think that the initial solution is simply translated in spacetime along suitable directions.

What are these directions?

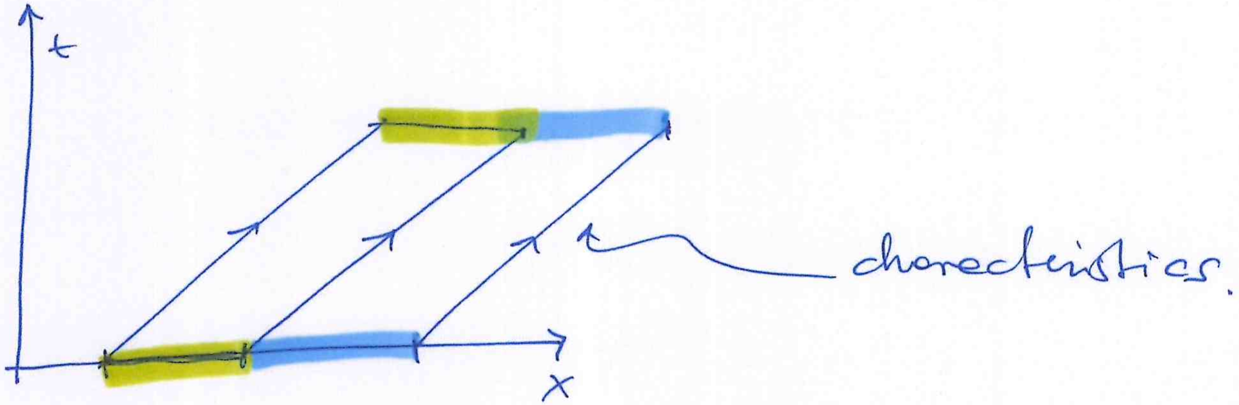
$$d_t u = \frac{du}{dt} = \partial_t u + \frac{dx}{dt} \partial_x u = 0 \quad \text{if} \quad \frac{dx}{dt} = \lambda = v = \text{const.}$$

the direction (straight line) $\frac{dx}{dt} = \lambda = v$ is called <u>characteristic</u> direction and it corresponds with the direction in spacetime along which the solution is propagated.

Eg
$u_0 = u_0(x) =$
$\sin(4\pi x)+1$



Because the solution is transported everywhere with the same velocity (ie along characteristic curves that are parallel) the solution is not distorted. This is all very clear and familiar.



We can generalize these results to a system of <u>linear</u> hyperbolic equations

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = 0$$

where A is a matrix of <u>constant</u> coefficients. (ie the system is linear)

Under these conditions we can define the vector of characteristic variables

$$\underline{W} := R^{-1}\,\underline{U}$$

where R is the matrix of right eigenvectors. Multiplying (*) by R^{-1} we obtain

$$R^{-1}\,\partial_t \underline{U} + R^{-1} A\cdot\nabla\,\underline{U} = 0$$

the right eigenvectors are constant

Now $R^{-1}\,\partial_t \underline{U} = \partial_t\left(R^{-1}\underline{U}\right) = \partial_t \underline{W}$

and $R^{-1} A\cdot\nabla\underline{U} = R^{-1} A\cdot\nabla\left(R\underline{W}\right) = R^{-1} A R\cdot\nabla\,\underline{W} = \Lambda\cdot\nabla\,\underline{W}$

where $\Lambda = R^{-1} A R$: diagonal[⊙] matrix with constant coefficients.

[⊙] diagonal because hyperbolic system

Putting things together

$$\partial_t \underline{U} + A \cdot \nabla \underline{U} = 0 \iff \boxed{\partial_t \underline{W} + \Lambda \cdot \nabla \underline{W} = 0} \quad (**)$$

(**) are called <u>characteristic equations</u> and state that the characteristic vector is conserved along the directions given by the eigenvalues of A, ie

$$d\underline{W} = \partial_t \underline{W}\, dt + \partial_{\vec{x}} \underline{W}\, d\vec{x}$$

$$= \left(\partial_t W + \frac{d\vec{x}}{dt} \partial_{\vec{x}} \underline{W}\right) dt$$

$$\frac{d}{dt}\underline{W} = \partial_t W + \Lambda \frac{\partial \underline{W}}{\partial \vec{x}} = 0$$

along $\Lambda = \frac{\partial \vec{x}}{\partial t}$

Note: here $\vec{x}$ represents the spatial coordinates of an orbitrary coordinate system. and is a matix, like Λ is a matix.

Since Λ is a diagonal matrix with coefficients λ_i the characteristic vector $\underline{W}$ is conserved along the directions

$$\lambda_{(i)} = \frac{\partial}{\partial t} \vec{x}_{(i)}$$: characteristic curves (characteristics)

Note that (**) are N independent advection equations and hence the solution (or value of $\underline{W}$) at any given time can be computed from the initial solution, ie the solution at $t=0$.

$$\boxed{W^i(x^j, t) = W^i(x^j - \lambda_i t, 0)}$$

As a result, also the original state vector can be computed rather trivially as

$$(\square) \qquad \underline{U}(x^j, t) = \sum_{i=1}^{N} W^i (x^j, t) \underline{R}^{(i)} = \sum_{i=1}^{N} W^i (x^j - \lambda_i t, 0) \underline{R}^{(i)}$$

In other words, once $\underline{W}(x^j, 0)$ is known, $\underline{U}$ can be computed at any position in space and time.

This is a very powerful result, which is however restricted to linear problems as in this case the characteristics do not intersect and hence the expression $(\square)$ is not double values.

What happens therefore in the case of <u>nonlinear</u> hyperbolic systems?

Once again it is simpler to understand this by starting from a simple example.

The simplest nonlinear hyperbolic equation is offered by the <u>inviscid[⊙] Burgers equation</u>

$$\partial_t u + u \, \partial_x u = 0$$

clearly, in this case the matrix A in (*) has a non-constant coefficient $a_{11} = u(x,t)$: function of space and time!

As for the advection equation we can write

$$\frac{d}{dt} u = \partial_t u + \frac{dx}{dt} \partial_x u = 0$$

u is conserved along the direction $\lambda = \frac{dx}{dt} = u(x,t) = \text{const}$

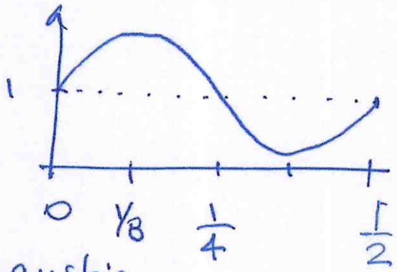
$$= u(x_0, 0) = u(x - \lambda t, 0)$$

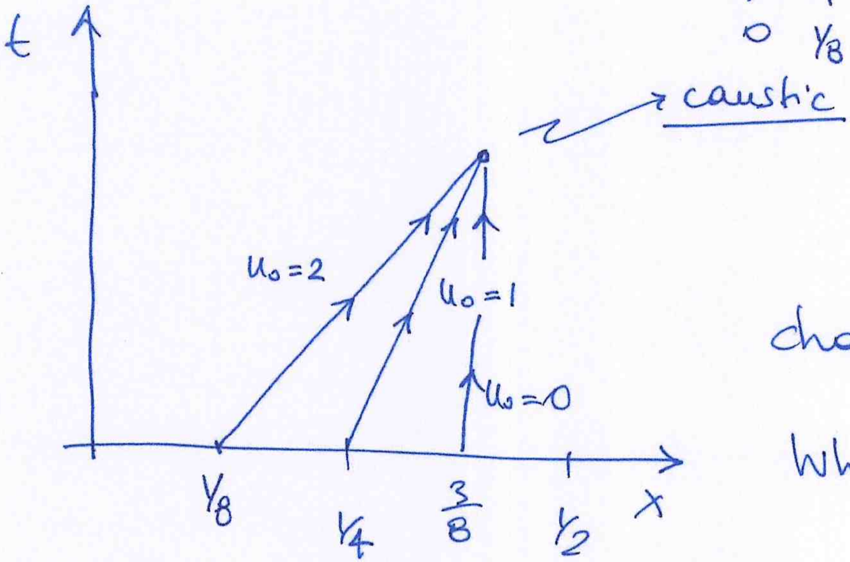
⊙ The (viscous) Burgers equation is given by $\partial_t u + u \partial_x u = \nu \, \partial^2_x u$

The characteristics are still straight lines, but they are no longer parallel!

Let's consider again the same initial data

$u_0 = \sin(4\pi x) + 1$





$$\frac{dx}{dt} = \lambda = u(x,t)$$

characteristics intersect!

When does this happen?

[one part of the solution moves faster than the other! Traffic.]

To calculate this we can write the implicit Burgers equation whose solution is

$$u(x,t) = u_0(x-\lambda t) = u_0(x - u(x,t)t)$$

Taking a time derivative yields ⊙

$$\partial_t u = [-u(x,t) - t\,\partial_t u]\,\partial_x u_0 \implies$$

$$\partial_t u(1 + t\,\partial_x u_0) = -u\,\partial_x u_0 \implies$$

$$\partial_t u = -\frac{u\,\partial_x u_0}{1 + t\,\partial_x u_0}$$; a caustic will be formed when the RHS will diverge,

which will diverge for $t = -\dfrac{1}{\min(\partial_x u_0)} = \dfrac{1}{4\pi} \sim \dfrac{1}{0.08}$

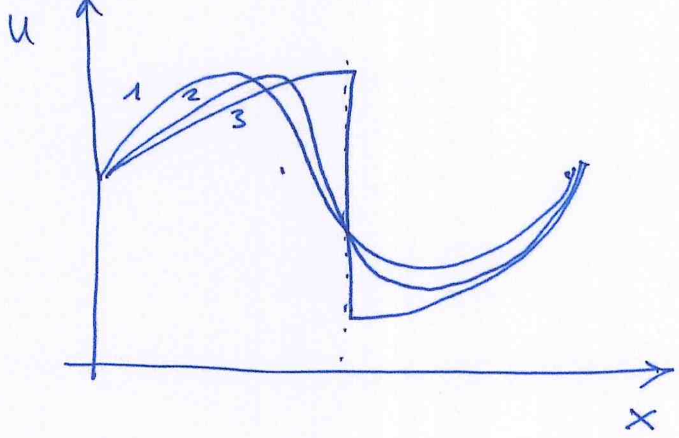
$\min(4\pi\cos(4\pi x)) = -4\pi$

-1

⊙ $\partial_t u_0 = \partial_x u_0\, \partial_t \dot{x} = \partial_x u_0\,(\partial_t(-ut))$

$= \partial_x u_0\,(-u - t\,\partial_t u)$

How does the solution change?



what is this?

This is a discontinuity or <u>shock</u>! The solution is mathematically discontinuous (double valued.)

In other words, the development of a caustic is equivalent to the development of a discontinuity or shock!

This process is called "wave steepening" and is typical of nonlinear hyperbolic problems.

Note that the wave steepening is unavoidable and occurs also from data that is initially smooth.

The importance of the RP stems from the fact that the solution of such a problem is at the hearth of advanced numerical scheme, named high-resolution shock-capturing schemes (HRSC). We will see these methods in the future lectures, but before then we need to discuss two important concepts: 1) <u>weak formulation</u>
2) <u>conservative formulation</u>.

Let's go back to the fact that the eqs. of relativistic hydrodynamics can be witten in the generic first-order form

$$\partial_t \underline{u} + A \cdot \nabla \underline{u} = \underline{S} \qquad (*)$$

which is said to be <u>hyperbolic</u> if the matrix A is diagonalizable with a set of real eigenvalues and linearly independent right eigenvector $\underline{R}^{(i)}$

If the matrix A is the Jacobian of a flux vector $F(\underline{u})$, ie if

$$A = \frac{\partial F(\underline{u})}{\partial \underline{u}}$$

then the <u>homogeneous</u> system $(*)$ can be written as

$$\partial_t \underline{u} + \nabla F(\underline{u}) = 0 \qquad (\square)$$

which represents the <u>conservative formulation</u> of $(*)$ with $\underline{u}$ the vector of <u>conserved variables.</u>

With these definitions in hand we can now discuss two theorems.

- Theorem I (Lax-Wendroff, 1960)

  "Conservative numerical schemes[①], if convergent, do converge to the week solution of the problem"

- Theorem II (Hou-Le Floch, 1994)

  "Numerical schemes not written in a conservative form, do not converge to the correct solution if a shock ~~were~~ is present in the flow."

① A numerical scheme based on the conservative formulation of the equations.

In other words: • if CF is used, then we are guaranteed to converge to the correct solution;

• if CF is not used, then we are guaranteed to converge to the incorrect solution.

Example: take Burgers equation with discontinuous initial data

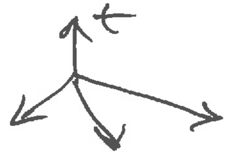
$$\partial_t u + u \partial_x u = 0$$

Any numerical solution will lead to incorrect propagation speed. However if Burgers eq. is rewritten in conservative form, ie

$$\partial_t u + \frac{1}{2} \partial_x (u^2) = 0$$

then the correct solution is obtained with any scheme.

• $\mathcal{M}$ 4D manifold with metric $\underline{g}$; space and time are indistinguishable but — it is convenient to split time away from space



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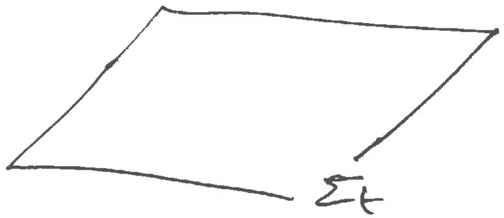


hypersurface at $t=$ const : fully spatial

• First, fix unit normal $\underline{n}$ such that $\underline{n}\cdot\underline{n}=-1$ and

$n_\mu \propto \nabla_\mu t = \Omega_\mu \Rightarrow \boxed{n_\mu = -\alpha \nabla_\mu t}$ $\underline{n}\cdot\tilde{\Omega} = \alpha^{-1}$: gradient of t is a function.

• Second, build projector $\perp$ to $\underline{n}$: $\underline{\gamma} = \underline{g} + \underline{n}\,\underline{n}$

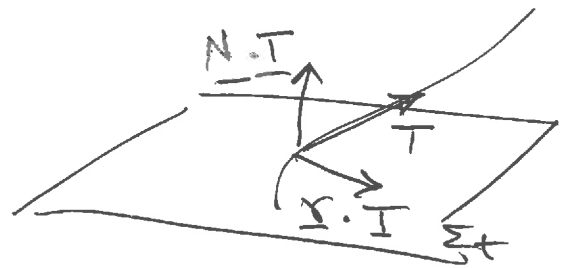
$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$: purely spatial $\underline{\gamma}\cdot\underline{n} = 0$

γ can be used to raise/lower indices but only for purely spatial tensors.

- Third, build projector along $\underline{n}$: $\underline{N} = -\underline{n}\,\underline{n}$; $N^\mu{}_\nu = -n^\mu n_\nu$

$\underline{N} \cdot \underline{\gamma} = 0$

- In this way we can split any tensor in a purely <u>spatial</u> and in a purely <u>time</u> part

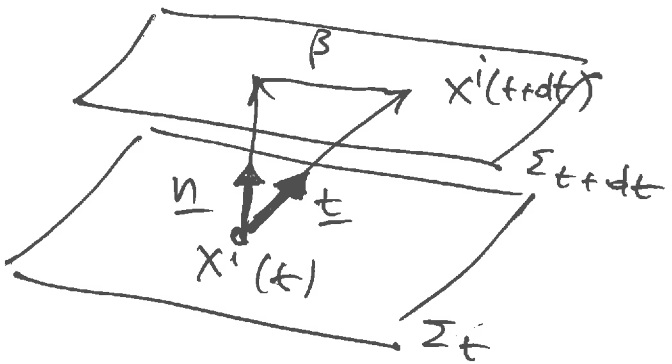


- $\underline{n}$ not parallel to $\tilde{\Omega}$; $\underline{n} \cdot \tilde{\Omega} = \alpha \neq$ const $\Rightarrow$

need 4-vector that is timelike and parallel to $\tilde{\Omega}$

$\underline{t} = \alpha \underline{n} + \underline{\beta}$ = (time part) + (space part) = $\underline{e}_t$

$\underline{t} \cdot \tilde{\Omega} = 1$



$x^i(t+dt) = x^i(t) - \beta^i(x^i, t)\,dt$

In this way we can build all the metric functions

Using the lapse and shift we can write the tt and ti parts of the metric as

$$g_{tt} = \underline{t}\cdot\underline{t} = -\alpha^2 + \beta^i\beta_i \qquad \left[(\alpha n^\mu + \beta^\mu)(\alpha n_\mu + \beta_\mu) = \alpha^2 n^\mu n_\mu + 2\alpha \cancel{n^\mu \beta_\mu} + \beta^\mu\beta_\mu\right]$$

$$g_{ti} = \underline{t}\cdot\underline{\gamma} = t^\mu \gamma_{\mu i} = t^\mu (g_{\mu i} + n_\mu \cancel{n_i}) = t_i =$$

$$\underset{\underline{t}\cdot\underline{n}=0}{=} (\alpha n^\mu + \beta^\mu) g_{\mu i} = \alpha n_i + \beta^\mu g_{\mu i}$$

$$= \alpha \cdot 0 + \beta^j \gamma_{ij}$$

$$= + \beta_i$$

so that the line element reads

$$ds^2 = -(\alpha^2 - \beta^i\beta_i)\,dt^2 + 2\beta_i\,dx^i dt + \gamma_{ij}\,dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -(\alpha^2 - \beta^i\beta_i) & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} ; \quad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

As a result

$$n_\mu = (-\alpha, 0, 0, 0) \; ; \qquad n^\mu = \frac{1}{\alpha}\left(1, -\beta^i\right)$$

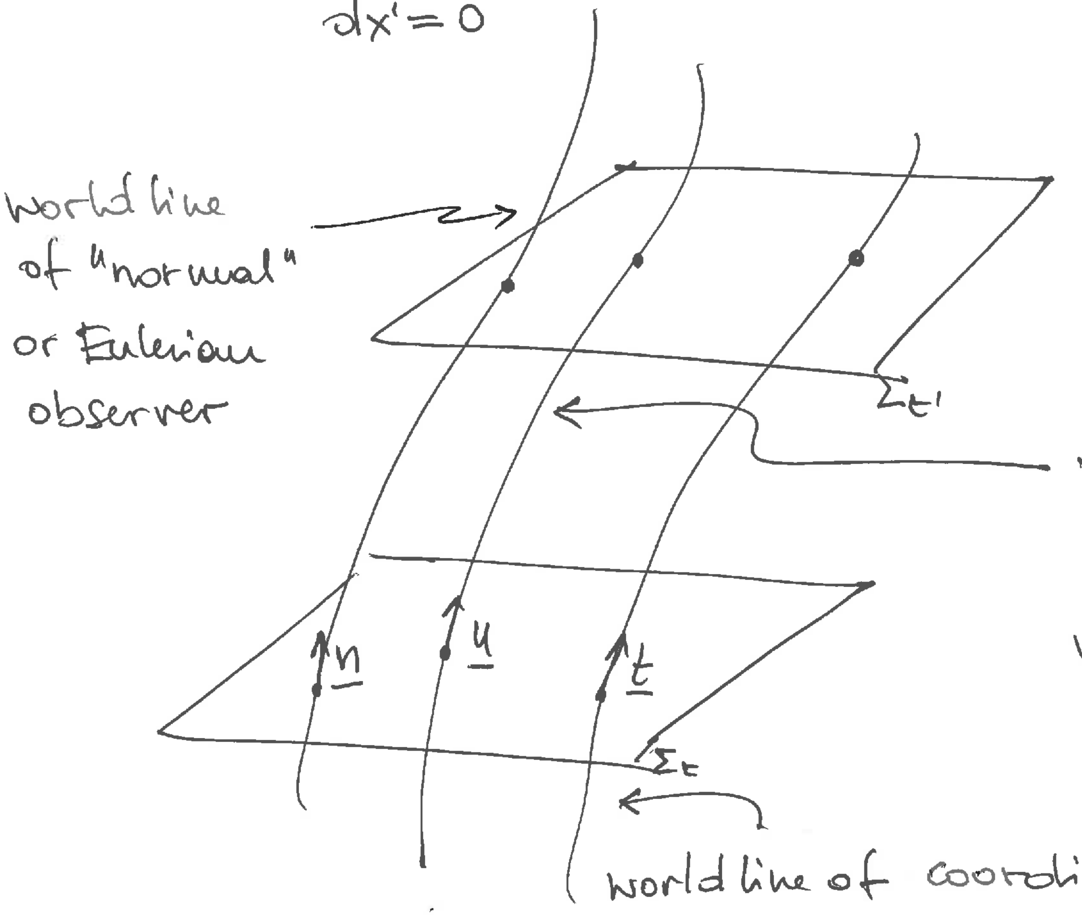
$$n^i = g^{\mu i} n_\mu = g^{0i} n_0 = -\alpha\left(\frac{\beta^i}{\alpha^2}\right) = -\beta^i/\alpha$$

$$n^0 = g^{00} n_0 = -\frac{1}{\alpha^2}\cdot(-\alpha) = \frac{1}{\alpha}$$

$$d\tau^2 = -ds^2 \underset{dx^i = 0}{=} +\alpha^2 dt^2$$

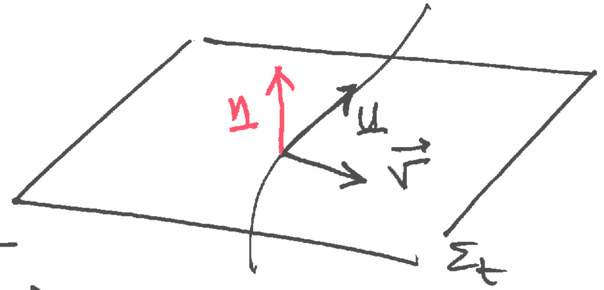
$$\underbrace{d\tau = \pm\alpha\, dt}$$

the lapse function expresses the rate of change of proper time relative to the change of coordinate time.



world line of fluid element with 4-velocity $\underline{u}$

An observer with tangent vector $\underline{n}$ is said to be a "normal" observer or "Eulerian" observer. This is the standard observer in a 3+1 split and it is relative to this observer that 3+1 quantities are measured. This is the case also for the fluid four-velocity $\underline{u}$



$$\vec{v}: \begin{pmatrix} \text{spatial part} \\ \text{of } \underline{u} \end{pmatrix} = \frac{(\text{projection of } \underline{u} \text{ on } \Sigma_t)}{(\text{projection of } \underline{u} \text{ along } \underline{n})}$$

$$= \frac{(\text{space})}{(\text{time})} = \frac{\gamma^i{}_\mu u^\mu}{-u_\mu n^\mu}$$

$W := -n_\mu u^\mu = \alpha u^t$: Lorentz factor

$W = (1 - v^i v_i)^{-1/2}$ as in special relativity (Exercise)

$W \to \infty$ for $v \to 1$

Prove $W = (1 - v^i v_i)^{-1/2}$

Proof

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right) \; ; \; v_i = \gamma_{ij} v^j = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^0} + \beta^j\right)$$

$$v^i v_i = \frac{1}{\alpha^2}\left[\left(\frac{u^i}{u^0} + \beta^i\right)\gamma_{ij}\left(\frac{u^j}{u^0} + \beta^j\right)\right] = \frac{1}{\alpha^2}\left[\gamma_{ij}\frac{u^i u^j}{(u^0)^2} + 2\beta^i \frac{u_i}{u^0} + \beta^i \beta_i\right] =$$

$$-1 = u^\mu u_\mu =$$

$$= g_{00}(u^0)^2 + 2 g_{0i} u^0 u^i + u^i u_i$$

$$= -(\alpha^2 - \beta^i \beta_i)(u^0)^2 + 2\beta_i u^i u^0 + u^i u_i \Rightarrow$$

$$\underbrace{\beta^i\beta_i (u^0)^2 + 2\beta_i u^i u^0 + u^i u_i}_{= -1 + (\alpha u^0)^2}$$

$$= \frac{1}{(\alpha u^0)^2}\left[u^i u_i + 2\beta^i u_i u^0 + (u^0)^2 \ldots\right]$$

$$= \frac{1}{(\alpha u^0)^2}\left(-1 + (\alpha u^0)^2\right) = \frac{-1 + W^2}{W^2} \rightarrow W^2(1 - v^i v_i) = 1 \Rightarrow$$

$$W = (1 - v^i v_i)^{-1/2} \quad \checkmark$$

In component form:

$$v^t = 0 \quad ; \quad v^i = \frac{\gamma^i{}_\mu u^\mu}{\alpha u^t} = \frac{(\delta^i_\mu + n^i n_\mu) u^\mu}{\alpha u^t} = \frac{u^i}{\alpha u^t} + \left(-\frac{\beta^i}{\alpha}\right) \frac{\overbrace{n_\mu u^\mu}^{-\alpha u^t}}{\alpha u^t}$$

$$= \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

$$v_t = 0 \quad ; \quad v_i = \frac{\gamma_{i\mu} u^\mu}{\alpha u^t} = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \frac{\gamma_{i0} u^0}{u^0}\right) = \frac{\gamma_{ij}}{\alpha}\left(\frac{u^j}{u^t} + \beta^j\right);$$

In other words, using $W = \alpha u^t$

$$\boxed{v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}} \qquad \boxed{v_i = \frac{u_i}{W} = \gamma_{ij}\left(\frac{u^j}{W} + \frac{\beta^j}{\alpha}\right)}$$

Note that $\underline{\gamma}$ does not lower indices of $\underline{u}$ (4D object)

I recall that in special relativity

$$v^i = \frac{u^i}{u^t} = \frac{dx^i/dt}{dt/d\tau} = \frac{dx^i}{d\tau}$$

Recap $v^i = \dfrac{u^i}{W} + \dfrac{\beta^i}{\alpha} = \dfrac{u^i}{\alpha u^t} + \dfrac{\beta^i}{\alpha}$

compare with special-relativistic expression

$$v^i_{SR} = \frac{u^i}{u^t} = \frac{u^i}{\gamma}$$

Hence, the three-velocity gains a dependence on α and $\underline{\beta}$

$v^i = v^i_{SR}$ if $\alpha = 1$; $\underline{\beta} = 0$

It's easy to show that

$$\underline{u} = W(\underbrace{\underline{n}}_{\text{purely time part}} + \underbrace{\underline{v}}_{\text{purely spatial part}})$$

$\Big(W = -\underline{n}\cdot\underline{u} \Rightarrow \underline{n}W = \underline{u}$

$u^i = Wv^i - \dfrac{\beta^i}{\alpha} W = W n^i + W v^i$

$u^0 = W n^0 = +\dfrac{1}{\alpha} W \Big)$

A conservative formulation can be derived also for the equations of relativistic hydrodynamics. To see how this is possible, let's start from the continuity eq. (conservation of rest mass)

$$0 = \nabla_\mu (\rho u^\mu) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \rho u^\mu)$$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t (\sqrt{-g}\, \rho u^t) + \partial_i (\sqrt{-g}\, \rho u^i) \right]$$

$\sqrt{-g} = \alpha \sqrt{\gamma}$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t (\sqrt{\gamma}\, \alpha \rho u^t) + \partial_i (\sqrt{\gamma}\, \alpha \rho u^i) \right]$$

$W = \alpha u^t$

$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$

$$= \frac{1}{\sqrt{-g}} \left[\partial_t (\sqrt{\gamma}\, \rho W) + \partial_i \left(\sqrt{\gamma}\, \alpha \rho \left(W v^i - u^t \beta^i\right)\right) \right]$$

Define $D := \rho W = \rho \alpha u^t$

$$\alpha \rho (W v^i - u^t \beta^i) = \alpha D v^i - \alpha D \beta^i = D(\alpha v^i - \beta^i)$$

$\Longrightarrow$

$$\boxed{0 = \partial_t (\sqrt{\gamma} D) + \partial_i [\sqrt{\gamma} D (\alpha v^i - \beta^i)]} \qquad (1)$$

D is a conserved quantity.

We can proceed in a similar way also for the other equations.

Recall:

$E := n_\mu n_\nu T^{\mu\nu}$: Eulerian energy density

$S^\mu := -\gamma^\mu{}_\alpha n_\beta T^{\alpha\beta}$: momentum density

$S^{\mu\nu} := \gamma^\mu{}_\alpha \gamma^\nu{}_\beta T^{\alpha\beta}$: spatial part of $T^{\mu\nu}$

$$T^{\mu\nu} = (e+p)\,u^\mu u^\nu + p\,g^{\mu\nu} = \rho h\,u^\mu u^\nu + p\,g^{\mu\nu}$$

$$= E\,n^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu} \qquad (*)$$

$u^\mu = W(n^\mu + v^\mu)$

$g^{\mu\nu} = \gamma^{\mu\nu} - n^\mu n^\nu$

$$= \rho h W^2 (n^\mu + v^\mu)(n^\nu + v^\nu) + p(\gamma^{\mu\nu} - n^\mu n^\nu)$$

$$= \rho h W^2 \left[n^\mu n^\nu + n^\mu v^\nu + v^\mu n^\nu + v^\mu v^\nu\right] + p(\gamma^{\mu\nu} - n^\mu n^\nu)$$

$\Longrightarrow$

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p\,\gamma^{\mu\nu}$$

$$S^\mu = \rho h W^2 v^\mu$$

$$E = \rho h W^2 - p$$

Next, we write the conservation of energy and momentum as

$$0 = \nabla_\mu T^{\mu\nu} = g^{\nu\lambda}\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu}{}_{\lambda}\right) - \frac{1}{2}T^{\alpha\beta}\partial_\lambda g_{\alpha\beta}\right] \Rightarrow$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu}{}_{\nu}\right) = \frac{1}{2}T^{\mu\lambda}\partial_\nu g_{\mu\lambda} \qquad (**)$$

We can now use (*) in (**) and restrict to a spatial index: $\nu \to j$ to obtain (Exercise)

$$(2) \quad \partial_t\left(\sqrt{\gamma}\,S_j\right) + \partial_i\left(\sqrt{\gamma}\left(\alpha S^i_j - \beta^i S_j\right)\right) = \frac{1}{2}T^{\mu\nu}\partial_j g_{\mu\nu}$$

$$= \sqrt{-g}\left(\frac{1}{2}S^{ik}\partial_j \gamma_{ik} + \frac{1}{\alpha}S_i\partial_j\beta^i - E\,\partial_j \ln\alpha\right)$$

Similarly, $n_\nu \nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu\left(n_\nu T^{\mu\nu}\right) - T^{\mu\nu}\nabla_\mu n_\nu$

from which we obtain (Exercise)

$$(3) \qquad \partial_t\left(\sqrt{\gamma}E\right)+\partial_i\left(\sqrt{\gamma}\left(\alpha S^i-\beta^i E\right)\right)=-\sqrt{-g}\,T^{\mu\nu}\nabla_\mu n_\nu$$

$$=\sqrt{-g}\left(K_{ij}S^{ij}-S^i\partial_i\ln\alpha\right)$$

Eqs (1)-(3) can be written in matrix form as

$$(4) \qquad \boxed{\partial_t\left(\sqrt{\gamma}\,\underline{U}\right)+\partial_i\left(\sqrt{\gamma}\,\underline{F}^i\right)=\underline{S}}$$

: flux balanced form

where

$$\underline{U}=\underbrace{\begin{pmatrix}\rho W\\ \rho h W^2 v_j\\ \rho h W^2-p\end{pmatrix}}_{\text{CONSERVED VARIABLES}} \; ; \quad \underline{F}^i=\begin{pmatrix}\alpha v^i D-\beta^i D\\ \alpha S^i{}_j-\beta^i S_j\\ \alpha S^i-\beta^i E\end{pmatrix}$$

$$\underline{U}=\left(D, S_j, E\right)^T$$

$$\underline{S} = \begin{pmatrix} 0 \\ \frac{1}{2}\alpha S^{ik}\partial_j \gamma_{ik} + S_i \partial_j \beta^i - E\partial_j \alpha \\ \alpha S^{ij} K_{ij} - S^j \partial_j \alpha \end{pmatrix}$$

(4) is called the "Valencia" formulation of the relativistic hydrodynamics equations. Valid in any 3+1 spacetime.

Notes

- E, D conserved variables $\Longrightarrow$

  $\tau := E - D = \rho W(hW - 1) - p$: also conserved variable

- ρ, v^i, ϵ: PRIMITIVE VARIABLES

$$
\begin{tikzcd}
\text{Primitive} \arrow[r, bend left, "\text{analytic}"] & \text{Conserved} \arrow[l, bend left, "\substack{\text{numerical} \\ \text{(root finding)}}"]
\end{tikzcd}
$$

The conversion of the conserved to primitive variables is a rather expensive operation that needs to be performed at each grid point via a nonlinear root finding algorithm. Optimizing this process is important and delicate.

Writing the hydrodynamic eqs in the conservative form (4) is important in those numerical methods that exploit the characteristic structure of the equations. In particular, it is straightforward to derive the

Jacobians $\quad A^{(i)} = \dfrac{\partial(\sqrt{\gamma}\,\underline{F}^{i})}{\partial(\sqrt{\gamma}\,\underline{U})} = \dfrac{\partial \underline{F}^{i}}{\partial \underline{U}}$

$A^{(i)}$ are 3 different $\underbrace{5 \times 5}_{\text{5 conserved quantities}}$ matrix; as an example

$i = 1$, then the eigenvalues of $A^{(x)}$ are

$\lambda_0 = \alpha v^x - \beta^x$ (triple eigenvalue; advective or entropy or matter wave)

$$\lambda_\pm = \frac{\alpha}{1 - v^2 c_s^2}\left\{ v^x (1 - c_s^2) \pm c_s \sqrt{(1-v^2)\left[\gamma^{xx}(1 - v^2 c_s^2) - v^x v^x (1 - c_s^2)\right.} \right\} - \beta^x$$

(acoustic waves).

In the limit of flat spacetime they reduce to

$\lambda_0 = v^x$

$\lambda_0 = \dfrac{v^x \pm c_s}{1 \pm v^x c_s} \sim v^x \pm c_s$ qed. $\quad$ Newtonien

See book for more info on ...

Linear waves

Such a fluid is described by an energy-momentum tensor we have already encountered

$$T_{\mu\nu} = (e+p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

$$= h e\, u_\mu u_\nu + p\, g_{\mu\nu}$$

Let's consider therefore the equations of conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0$$

which are given by $(\mu = 0, 1)$

$$\begin{cases} \partial_t \left[(e + p v^2) W^2 \right] + \partial_x \left[(e+p) W^2 v \right] = 0 \\ \partial_t \left[(e+p) W^2 v \right] + \partial_x \left[(e v^2 + p) W^2 \right] = 0 \end{cases}$$

Exercise

where $\quad u^\mu = W(1, v) \;; \quad W = (1-v^2)^{-1/2}$

Let now e_0, p_0 and $v_0 = 0$ be the values of the energy density, pressure and velocity (fluid at rest) and introduce first-order perturbations of the type

$$e = e_0 + \delta e\,, \quad p = p_0 + \delta p\,, \quad v = v_0 + \delta v = \delta v\,.$$

The resulting set of perturbation equations will be

$$\begin{cases} \partial_t (\delta e) + (e_0 + p_0)\, \partial_x \delta v = 0 \\[2ex] \partial_t \delta v + \dfrac{1}{e_0 + p_0}\, \partial_x \delta p = 0 \end{cases}$$

where we have used $\left.\begin{array}{l} \partial_t e_0 = 0 = \partial_x e_0 \\ \partial_t p_0 = 0 = \partial_x p_0 \end{array}\right\}$ stationary and uniform flow.

Taking an additional time derivative and combining terms we obtain

$$(*) \qquad \partial_t^2 \delta e - \frac{\delta p}{\delta e} \partial_x^2 \delta e = 0 \iff \Box \delta e = 0$$

Similarly $\begin{cases} \Box \delta e = 0 \\ \Box \delta p = 0 \end{cases}$: wave equations with speed c_s.

$$c_s^2 = \delta p / \delta e$$

Eqs (*) show that perturbations propagate as waves with speed c_s. These are $\underbrace{\text{acoustic waves}}_{\text{(or sound waves)}}$ and c_s is the sound speed. More generally

$$c_s^2 = (\partial p / \partial e)_s \qquad s: \text{specific entropy}$$

The linearization approach has clearly removed all the nonlinearities, but as mentioned before, the hydrodynamic equations are intrinsically nonlinear and generically lead to nonlinear waves. These can be distinguished as follows:

- simple waves

are the nonlinear equivalent of sound waves but solutions of the full nonlinear eqs. They are always associated to a single eigenvalue, for which some quantities of the flow (called Riemann invariants) are conserved (see book for details). A theorem by Friedrichs states that: "any one-dimensional smooth solution neighbouring a constant state must be a simple wave". Examples of simple waves are rarefaction/compression waves

- <u>discontinuous waves</u>

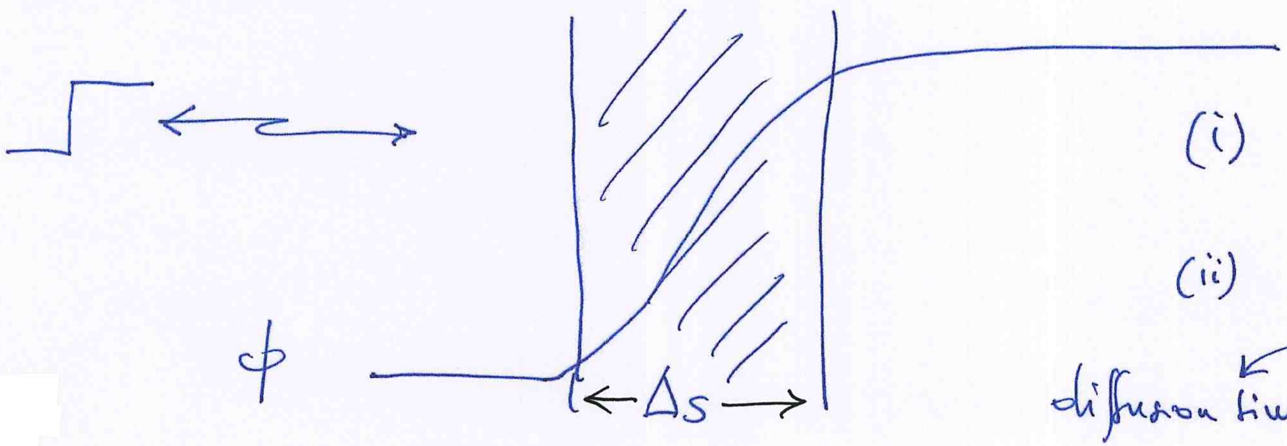
are regions of the flow in which some of the fluid properties (e.g., velocity, rest-mass density, etc) are <u>taken</u> to be discontinuous. More on this later. They can be further distinguished into

  - <u>contact waves</u>: surfaces seperating two parts of the flow with different properties but without flow through the surface (contact discontinuities)
  - <u>shock waves</u>: same as above but with flow across the surface; fluid on either side have different properties but no chemical/physical change (density, energy, etc) takes place across surface.

As discussed when considering the Burgers equation discontinuous waves can be produced from compressive motions having smooth initial data.

A discontinuous wave is a mathematical artefact to cope with flows in which the properties vary very rapidly on a very small lengthscale. No physics breaks down at a shock wave! Simply, complex and steep gradients are replaced by simple junction conditions.

The use of shock wave is reasonable when



lengthscale of variation

(i) $\Delta s \approx l_{mfp} \ll \phi / \partial_x \phi$

(ii) $\tau \ll \phi / \partial_t \phi$

diffusion timescale

timescale of front motion

If these conditions are not met, then the discontinuous wave approximation is not valid and more sophisticated approaches are necessary, eg Boltzmann equation.

The junction conditions mentioned above are simple algebraic conditions that guarantee the conservation across the shock front of rest-mass, energy and momentum.

To derive these conditions in a covariant form we start from the conservation equations

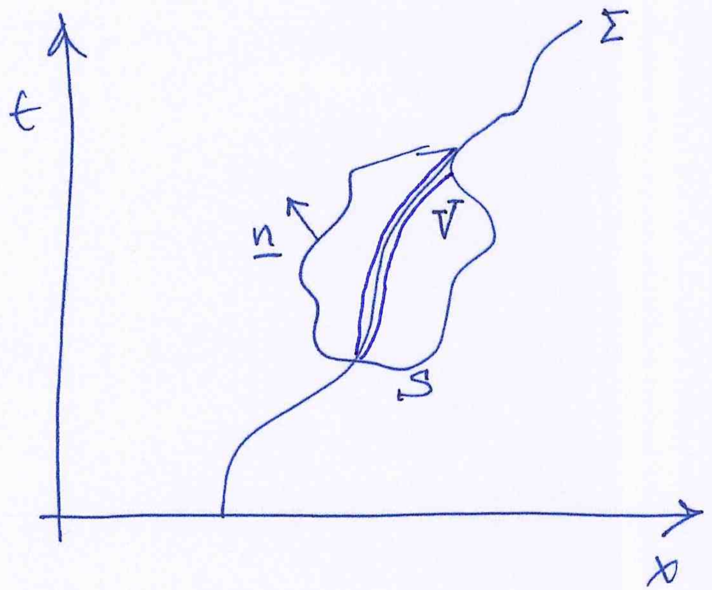
$$\begin{cases} \nabla_\mu (\rho u^\mu) = 0 \\ \nabla_\mu T^{\mu\nu} = 0 \end{cases}$$

which we rewrite as

$$\begin{cases} \nabla_\mu (\rho u^\mu f) = \rho u^\mu \nabla_\mu f \\ \nabla_\mu (T^{\mu\nu} \lambda_\nu) = T^{\mu\nu} \nabla_\mu \lambda_\nu \end{cases}$$

where f and λ_ν are an arbitrary scalar function and vector field, respectively

Let Σ be the history of a 2D spacelike surface representing the shock front. Let $\mathcal{V}$ be a 4D volume around Σ.



$$\int_{\mathcal{V}} \nabla_\mu (\rho u^\mu f) d^4x = \int_{\mathcal{V}} \rho u^\mu \nabla_\mu f \, d^4x$$

$$= \oint_{\mathcal{S}} \rho u^\mu f n_\mu \, d^3x$$

Stoke's theorem

Similarly

$$\int_{\mathcal{V}} \nabla_\mu (T^{\mu\nu} \lambda_\nu) d^4x = \int_{\mathcal{V}} T^{\mu\nu} \nabla_\mu \lambda_\nu \, d^4x = \oint_{\mathcal{S}} T^{\mu\nu} \lambda_\nu n_\mu \, d^3x$$

Consider now the limit in which $\mathcal{V} \to 0$. The first two integrals vanish, while the third ones reduce to the calculation of the integrand on both sides of Σ, ie

$$
(*) \quad \begin{cases} \displaystyle\int_{\Sigma'} f \, [\![\rho u^\mu]\!] \, n_\mu \, d^3x = 0 \\[2ex] \displaystyle\int_{\Sigma'} \lambda_\mu \, [\![T^{\mu\nu}]\!] \, n_\nu \, d^3x = 0 \end{cases}
$$

ahead ("1")

behind ("2")

where $\displaystyle [\![Q]\!] \equiv \underbrace{Q_a - Q_b}_{\text{jump of } Q \text{ across the shock front.}}$: "double bracket notation"

Given the arbitrariness in the choice of f and λ, the conditions $(*)$ can be satisfied iff

$$
(**) \quad \begin{cases} [\![\rho u^\mu]\!] \, n_\mu = 0 \\[2ex] [\![T^{\mu\nu}]\!] \, n_\nu = 0 \end{cases}
$$

: (relativistic) Rankine Hugoniot conditions (junction / jump conditions)

Riemann problem

Determine the flow pattern (ie the number and type) of nonlinear waves that develops from constant and discontinuous initial data.

Riemann worked on this more than 150 years ago and the solution of this problem is the basis for many advanced numerical methods in (relativistic) hydrodynamics

Mathematically, this is defined as

$$\underline{U}(x,0) = \begin{cases} \underline{U}_L & \text{if } x<0 \\ \underline{U}_R & \text{if } x>0 \end{cases} \qquad \begin{array}{l} \underline{U}_L \neq \underline{U}_R \\ \underline{U}_L = \text{const}; \\ \underline{U}_R = \text{const}; \end{array}$$

Physically you can think of a tube containing a membrane and in which you can specify the properties of the fluid on either side of the membrane

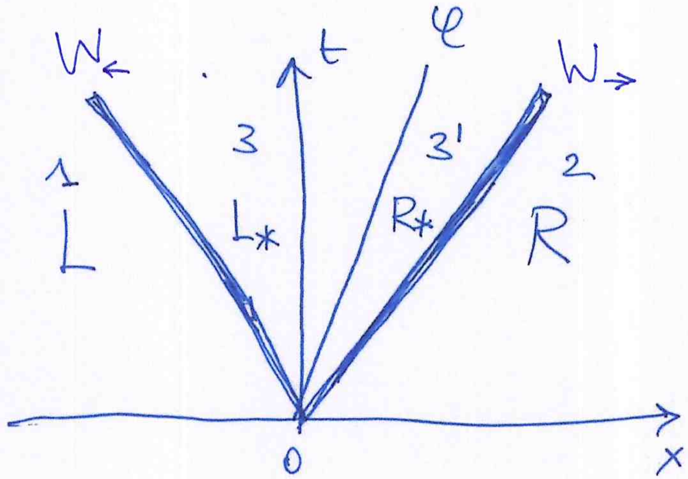
The problem consists then in determining the evolution of the system if the membrane is removed istantaneously.

This solution is schematically represented as

$$L \; R \qquad \text{at } t=0$$

$$L \; \overleftarrow{W} \; L_* \; C \; R_* \; \overrightarrow{W} \; R \qquad \text{at } t>0$$

- $\overleftarrow{W}$: left propagating nonlinear wave
- C: contact discontinuity
- $\overrightarrow{W}$: right-propagating nonlin. wave



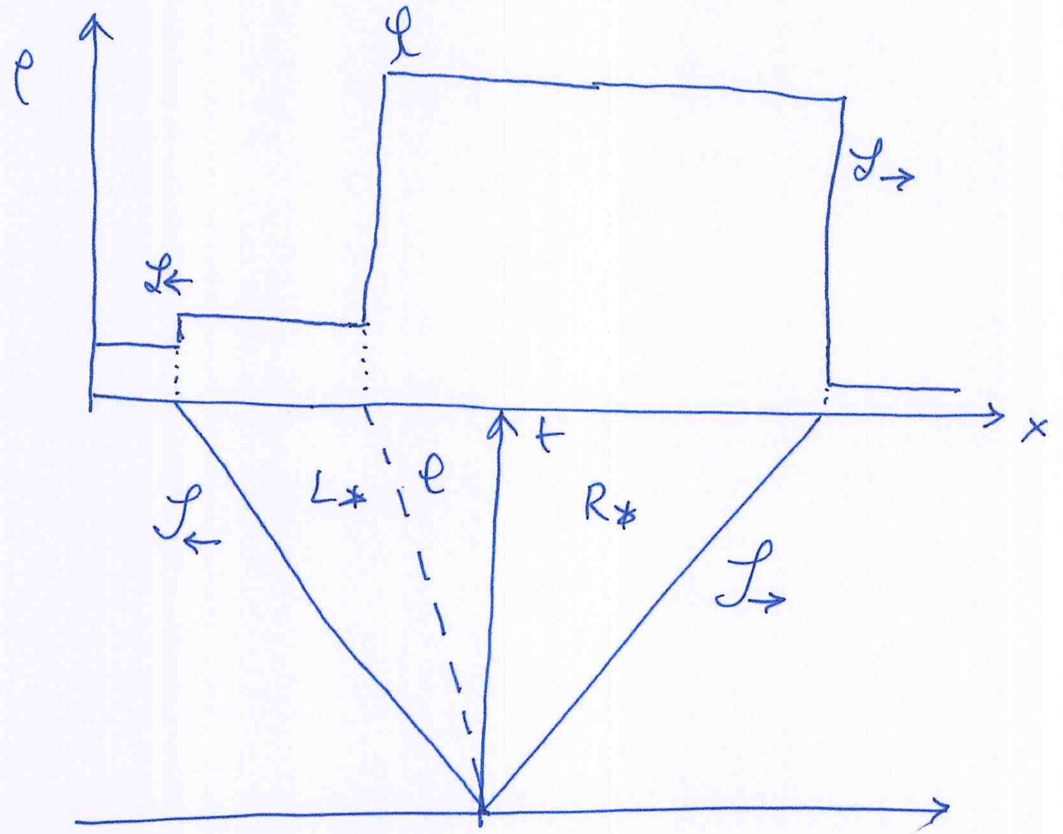
Note:

- the states L and R are the original ones as the waves have not yet reached them
- the regions $L*$ and $R*$ are separated by a contact discontinuity and hence $p_{L*} = p_{R*} = p_*$
- no assumption is made on the waves $W_{\rightarrow}$ and $W_{\leftarrow}$: these can be shocks or rarefaction waves.

In Newtonian hydrodynamics, Riemann concluded that the one-dimensional flow resulting from the initial data (*) will lead to four different solutions, or equivalently, three wave patterns.

(i) two shock waves moving to the right and to the left

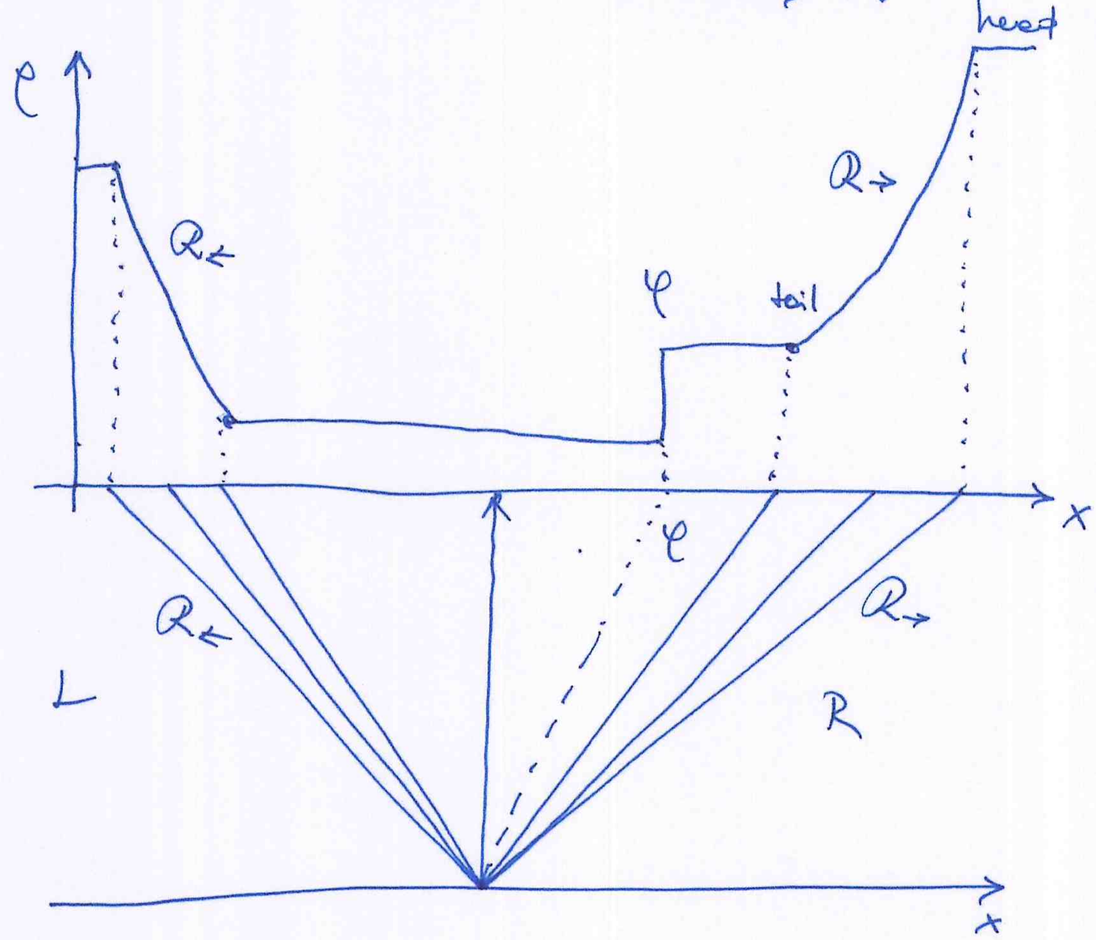
$$L \ \mathcal{S}_{\leftarrow} \ L_* \ \mathcal{C} \ R_* \ \mathcal{S}_{\rightarrow} \ R$$



Note that the shocked fluid has larger density.

(ii) two rarefaction waves: moving to the right and to the left

$$L \; \mathcal{R}_{\leftarrow} \; L_* \; \mathcal{C} \; R_* \; \mathcal{R}_{\rightarrow} \; R$$



(iii) one shock and one rarefaction wave: moving to the right/left

$$L \quad \mathcal{R}_{\leftarrow} \quad L_* \quad \mathcal{C} \quad R_* \quad \mathcal{S}_{\rightarrow} \quad R$$

$$L \quad \mathcal{S}_{\leftarrow} \quad L_* \quad \mathcal{C} \quad R_* \quad \mathcal{R}_{\rightarrow} \quad R$$