1. The purpose of these lectures...

is to give an overview of GR & curved spacetime effects. I had to make choices and start from somewhere. I will assume that you have seen the field equations of GR, but this is not fundamental.

I also provide some (primitive) Mathematica notebooks, so that you can get acquainted with all calculations. I urge you to consider that the field equations are complicated! Any help that one can get is welcome...

Finally: abuse me & other speakers... ask for extra material, ask questions, LEARN!

References: ~~www.~~ blackholes.ist.utl.pt

- Poisson & Will: "Gravity: Newtonian, Post-Newtonian Relativistic"
- Chandrasekhar: the Mathematical theory of BHs
- MTW: "Gravitation"
- Shapiro & Teukolsky: "Black holes, White Dwarfs & neutron stars"
- LIGO LSC: "the basic physics of the binary black hole merger GW150914" arXiv:1608.01940

2. Newton's theory [~1687]

$$\vec{F} = m_i \vec{a}$$
$$\vec{F}_g = -\frac{GMm_g}{r^2}\vec{u}_r$$

Newton's theory has $m_i = m_g$ [he was aware of implications] and describes 300+ years of science.
In 1845, Le Verrier postulated the existence of a "dark-matter-type object" to explain anomalies in the orbit of Uranus. Neptune was discovered in 1846.
Le Verrier would propose the existence of Vulcano, years later, to explain anomalies in Mercury...

The two-body problem is solvable in Newtonian dynamics. Take a massive body and a small test mass.* there are two conserved quantities

$$L = mr^2\dot{\phi} \qquad \tilde{L} = r^2\dot{\phi}$$
$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \Phi(r) \qquad \tilde{E} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi$$
$$\Rightarrow \dot{r}^2 = 2\left[\tilde{E} - \frac{1}{2}\frac{\tilde{L}^2}{r^2} - \Phi\right] \equiv V(r)$$
$$2\dot{r}\ddot{r} = V'\dot{r} \Rightarrow \ddot{r} = \frac{V'}{2} = \tilde{L}^2 r^{-3} - \Phi'$$

For circular orbits, $V = V' = 0 \Rightarrow L^2 = r^3\Phi'$

for $\Phi = -\frac{GM}{r} \Rightarrow L^2 = rGM \Leftrightarrow v^2 = \frac{GM}{r}$

* the full problem is easily solvable as well

Suppose now that $r = r_c + \chi$, $\chi \ll r_c$

then, since $\ddot{r}_c = 0 = V'(r_c)$ we get

$$\ddot{\chi} = \frac{\chi}{2} V''(r_c) \Rightarrow \chi = e^{\sqrt{\frac{V''}{2}}\, t}$$

In other words, the orbit is stable if $V'' < 0$, it oscillates around r_c with frequency $\sqrt{-\frac{V''}{2}}$.

for $\Phi = -\frac{GM}{r^\alpha}$, we get

$$\frac{V'}{2} = \tilde{L}^2 r^{-3} - \alpha G M r^{-\alpha-1}$$

$$\frac{V''}{2} = -3\tilde{L}^2 r^{-4} + \alpha(\alpha+1) G M r^{-\alpha-2}$$

$$V'_{(r_c)} = 0 \Rightarrow \tilde{L}^2 = \alpha G M r^{2-\alpha}$$

$$\frac{V''}{2}(r_c) = -3\alpha G M r^{-\alpha-2} + \alpha(\alpha+1) G M r^{-\alpha-2}$$

$$= \alpha G M r^{-\alpha-2}\left[\alpha - 2\right]$$

thus only for $\alpha < 2$ are circular orbits stable*!

And this is it. We solved the two body problem for a specific situation. We can be more sophisticated, but the full problem admits a closed-form solution: generic bound motion is an ellipse.

One reason for this simplicity is that time is practically absent from Newton's theory... the gravitational interaction travels instantaneously

* for $\alpha = 2$ we need to look into V'''! Orbits are unstable in this case.

* And they close for $\alpha = 1$! $\left[\sqrt{\frac{V''}{2}} = \omega\right]$

Let us focus now on configurations describing finite-sized objects in isolation. The way to handle this is to write down the equations of motion:

1. Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0, \quad \begin{array}{l} v = \text{velocity} \\ \rho = \text{density} \end{array}$$

2. Momentum or Euler equation $\left(\vec{a} = \frac{\vec{F}}{m}\right)$

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\nabla P - \nabla \Phi$$

P is pressure

Φ is gravitational potential

3. Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho$$

You need 1, or else the physics would not be complete.

For objects in static equilibrium $\frac{\partial}{\partial t} \equiv 0$ and spherically symmetric $\left[\nabla = \frac{\partial}{\partial r}\right]$ we have

2.2. $\frac{dP}{dr} = -\rho \frac{d}{dr}\Phi$

3.2. $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\Phi\right) = 4\pi G\rho \Rightarrow \frac{d}{dr}\Phi = \frac{G}{r^2}\int 4\pi r^2 \rho\, dr$

Defining $m(r) = \int 4\pi r^2 \rho \, dr$ we have

$\frac{d}{dr}\Phi = \frac{G}{r^2} m(r)$, thus 2.2 gives

$$\begin{cases} \frac{dP}{dr} = -\frac{G\, m(r)\, \rho}{r^2} \\ \frac{dm}{dr} = 4\pi r^2 \rho \end{cases}$$

We miss something! What?! Well, we miss the small-scale physics of what is it that is creating the pressure... this is an important point, one that others will address more carefully. What we miss is the equation of state (EoS), relating P to ρ.
A well-known example is that of a perfect gas at constant temperature, for whic $P \propto \rho$

Let's solve the equations above with a simple EoS: $\rho = $ const. In this case, we get

$m = \frac{4}{3}\pi r^3 \rho_0$ and

$$\frac{dP}{dr} = -\frac{4}{3}\pi G \rho_0^2 r \Rightarrow P = \frac{2}{3}\pi G \rho_0^2 (R^2 - r^2)$$

The pressure is zero at the surface and maximum at the center. The star's mass is arbitrary!

A second useful example concerns a polytropic EoS, $P = K\rho^2$. Then, it's easy to show [HW] that

$$\rho = \rho_c \frac{\sin(\omega r)}{\omega r}, \quad \omega = 4\sqrt{\frac{\pi G}{8K}}$$

Notice that ρ_c is arbitrary, and $\omega R = \pi$. The total mass of the star, $M = \frac{4\pi^2 \rho_c}{\omega^3}$ is still arbitrary.

$\boxed{\text{Notebook Newtonian\_stars}}$

Suppose now that we want to understand if such configurations are stable. We then need to displace the configuration. The disturbances are characterized by a (Lagrangian) displacement $\xi(\vec{x},t)$ connecting fluid elements in the unperturbed state to fluid elements in the perturbed state.

Define the adiabatic index of the perturbations,

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho}$$

then [HW]

$$\ddot{\xi} - \frac{1}{\rho}\left[\frac{\gamma P}{r^2}\left(r^2 \xi\right)'\right]' + \frac{4}{\rho r} P' \xi = 0$$

We will need to solve this PDE. It is useful to Fourier decompose the field $\xi(\vec{x},t) = \int e^{i\omega t} \xi(\vec{x},\omega)\, d\omega$

and we find

$$-\omega^2 \xi - \frac{1}{\rho}\left[\frac{\gamma P}{r^2}\left(r^2 \xi\right)'\right]' + \frac{4}{\rho r} P' \xi = 0$$

[Notebook] Newtonian_stars

We can solve this equation for constant density stars, $P = b\left(R^2 - r^2\right)$

Defining $r = Rx$, we obtain

$$(1-x^2)\xi'' + \xi'\left[\frac{2}{x} - 4x\right] + \xi\left[\underbrace{\frac{8}{\gamma} + \frac{\rho\omega^2}{\gamma b} - 2}_{A} - \frac{2}{x^2}\right] = 0$$

this is an hypergeometric equation, which we can solve using a power series exponsion [frobenius]

Let

$$\xi = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad \text{and find}$$

$$a_{2n+1} = 0$$

$$\frac{a_{n+2}}{a_n} = \frac{n^2+5n+4-A}{n^2+7n+10}, \quad n = 0, 2, 4 \ldots$$

For the series to converge, $A = n^2+5n+4$ or

$$\omega^2 = \frac{2\pi G\rho}{3}\left[\gamma(n^2+5n+6) - 8\right], \quad n = 0, 2, 4, \ldots$$

- Instability for $\gamma < 4/3$
- Period of mHz for Earth and Sun-like

MOND (Modified Newtonian Dynamics)

There have been attempts at changing the form of Newton's law to explain ~~d~~ some observations, most notably dark matter. In a nutshell, stars at large distances follow a velocity profile

$v_r \propto$ const, instead of the usual $v \propto \frac{1}{\sqrt{r}}$ profile $\left[GM\frac{1}{r^2} \simeq \frac{v^2}{r}\right]$. This seems to imply that the force felt is not the usual gravitational force, or then that the $\vec{F} = m\vec{a}$ relation needs to be modified. Milgrom postulated that

$\vec{F} = m\,\mu\left(\frac{a}{a_0}\right)\vec{a}$, with $\mu(a/a_0)$ a function not known a priori, and a_0 a const marking the transition to the new regime.

for $\mu = \left(1 + \frac{a_0}{a}\right)^{-1}$, we get

$$\vec{F} = m\vec{a}, \quad a \gg a_0$$

$$\vec{F} = m\frac{a}{a_0}\vec{a}, \quad a \ll a_0$$

for circular orbits,

$$\frac{GMm}{r^2} = m\frac{\left(\frac{v^2}{r}\right)^2}{a_0} \Leftrightarrow \boxed{v^4 = GMa_0}$$

$a_0 \sim 1.2 \times 10^{-10}\, ms^{-2}$

Newtonian physics has problems
[Absolute time, instantaneous propagation]

Einstein had already solved some with SR:

Introduce ST which consists of events each requiring 4 number for complete ~~spp~~ specification

Observers make measurements, i.e., assign coordinates to events: observer is a choice of coordinates

SR has ~~special set~~ preferred family of observer Inertial observers, for whom free particles move with uniform velocity.

For any Lorentz frame, interval between nearby events is

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

with dt, dx, dy and dz are the differences between the coordinates of the events in any Lorentz frame.

If $x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$ we can write

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \quad , \quad \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$$

$\eta_{\alpha\beta} \downarrow$ metric tensor

One can use other coordinates, for example non-inertial polar, where

$$ds^2 = g_{\alpha\beta}(y^\gamma)\, dy^\alpha dy^\beta$$

ST is still flat: there is change of coordinates that put metric in Minkowskian form <u>everywhere</u>

Einstein leaned on the <u>equivalence</u> principle to understand gravity. He elevated spacetime to curved manifold

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

where <u>no</u> choice of coordinates reduce the metric to Minkowski everywhere.

<u>BUT</u> spacetime is locally flat, can always find a local transformation that does that.

In fact, it is possible to find a locally inertial frame where

$$ds^2 = \left[\eta_{\alpha\beta} + \mathcal{O}(|x|^2)\right] dx^\alpha dx^\beta$$

thus, in GR acceleration can always be made to disappear, by going to a freely falling coordinate sistem. <u>BUT</u> the difference between gravitational accelerations of two nearby bodies is not removal in general.

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} \cdot \frac{G}{c^4}$$ → geometric units $G = c = 1$

In these, $M_\odot = 1.4766\,\mathrm{Km}$
$= 4.92 \times 10^{-6}$ secs

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

$$R_{\mu\nu} = R^{\eta}{}_{\mu\eta\nu}$$

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\nu} - \Gamma^{\mu}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\nu}$$

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha}\left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}\right)$$

[Notebook] Field_Equations

- L.H.S. is built purely from $g_{\mu\nu}$, it's pure geometry the r.h.s is matter. Any form of energy gravitates ($E = mc^2$)

- These E.O.M. also follow from Einstein-Hilbert Lagrangian. In fact [Lovelock]:

In 4-dimensions, the only divergence-free symmetric Rank-2 tensor constructed solely from $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance*, is the Einstein tensor plus a cosmological constant.

- Special features:
$D_\mu T^{\mu\nu} = 0$ [geodesic motion & WEP] → in Absence of GWs!

- Reduces to Poisson equation in Newtonian limit [small v, small grav. potencial]

$*$ ≡ coordinate reparametrizations

Spherically symmetric solutions

Spherical symmetry $\Rightarrow$ there is a coordinate system (t, r, θ, φ) in which line element is invariant under reflections

$\theta \to \theta' = \pi - \theta$

$\varphi \to \varphi' = -\varphi$

In other words, no cross terms $dr\,d\theta$, $dr\,d\varphi$, $d\theta\,d\varphi$, $dt\,d\theta$, $dt\,d\varphi$ are allowed. In addition, each 2D submanifold defined by $t = \text{const}$, $r = \text{const}$, are the 2-spheres

$$dl^2 = a^2 (d\theta^2 + \sin^2 d\phi^2) = a^2 d\Omega_2^2$$

therefore,

$$ds^2 = -A(t,r)dt^2 + 2B(t,r)dt\,dr + C(t,r)dr^2 + D(t,r)\,d\Omega_2^2$$

changing the radial coordinate $r \to \tilde{r} = \sqrt{D}$

$$ds^2 = -A\,dt^2 + 2B\,dt\,d\tilde{r} + C\,d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2$$

and introducing a new time coordinate

$$d\tilde{t} = I(t, \tilde{r}) \left[-A\,dt + B\,d\tilde{r}\right]$$

$$d\tilde{t}^2 = I \quad \left[A^2 dt^2 - 2AB\,dt\,d\tilde{r} + B^2 d\tilde{r}^2\right]$$

$$\Rightarrow -A\,dt^2 + 2B\,dt\,d\tilde{r} = -\frac{d\tilde{t}^2}{I^2 A} + \frac{B^2}{A} d\tilde{r}^2$$

this means that the most general, spherically symmetric line element can be written as

$$ds^2 = -e^{2\Phi(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 d\Omega^2$$

or

$$ds^2 = -f(t,r)\,dt^2 + \frac{1}{g}\,dr^2 + r^2 d\Omega^2$$

Motion of test particles in GR

Test particles are idealizations: objects moving freely in the gravitational field

- In SR, one can get their equation of motion from a variational principle that extremizes the interval along the worldline:

$$\delta \int ds = 0 \quad \text{with Lagrangian } L = \left(-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{1/2}$$

Proof: $ds = \left(-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)^{1/2} d\lambda$, with $\dot{x}^\alpha \equiv \dfrac{dx^\alpha}{d\lambda}$

the Euler-Lagrange equations are

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}, \quad \text{now}$$

$$\begin{cases} \partial L / \partial x^\alpha = 0 \\ \dfrac{\partial L}{\partial \dot{x}^\alpha} = -\dfrac{\dot{x}^\beta}{L}\eta_{\alpha\beta} \end{cases} \Rightarrow \eta_{\alpha\beta}\ddot{x}^\beta - \frac{1}{L}\frac{dL}{d\lambda}\eta_{\alpha\beta}\dot{x}^\beta = 0$$

If we choose λ such that $L = \text{const}$, then λ is an affine parameter. Let us choose $\lambda = s$ in whic case $L = 1$ [because $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$]

$$\Rightarrow \eta_{\alpha\beta}\ddot{x}^\beta = 0 \Rightarrow \ddot{x}^\gamma = 0$$

this is the well-known equation for uniform velocity this curve is a geodesic in flat spacetime, as it extremizes the length.

the equivalence principle then says that there must be a variational principle for the motion of test particles in GR: free particles move along geodesics of spacetime but now

$$L = \left[-g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\right]^{1/2}$$

Going through the same calculation, the Euler-Lagrange

$$g_{\alpha\beta}\ddot{x}^\beta + g_{\alpha\beta,\gamma}\dot{x}^\gamma \dot{x}^\beta - \frac{1}{2} g_{\gamma\beta,\alpha}\dot{x}^\gamma \dot{x}^\beta = 0,$$ where we used

$$\frac{d}{d\lambda} g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\gamma}{d\lambda}, \quad \text{and} \quad g_{\alpha\beta,\gamma} \equiv \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$$

and we assumed $L = \text{const}$; but

$$g_{\alpha\beta,\gamma}\dot{x}^\beta \dot{x}^\gamma = \frac{1}{2}\left[g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta}\right]\dot{x}^\beta \dot{x}^\gamma$$

thus

$$g_{\alpha\beta}\ddot{x}^\beta + \Gamma_{\alpha\beta}\,\dot{x}^\beta \dot{x}^\gamma = 0,$$ where Γ are the Christoffel

$$\Gamma_{\alpha\beta\gamma} \equiv \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}\right)$$

We can write this as

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0$$

Notice how the equivalence principle is satisfied: in a local inertial frame one can choose $g_{\alpha\beta,\gamma} = 0$ $\Rightarrow \Gamma$ vanishes (and we recover SR).

The way this works in the field equations is simple: the Lagrangian corresponds to the tensor

$$T^{\mu\nu} = mc\int u^\mu u^\nu \frac{\delta(x^\mu - r^\mu(\tau))}{\sqrt{-g}}\,d\tau$$

the divergence of which implies geodesy!

Recall that $L \propto g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta}$

$$= -f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 ; \quad f \equiv 1 - \frac{2M}{r}$$

Lagrangian does not depend on (t, φ), therefore on the equatorial plane we have

$$f\dot{t} = E \quad \text{and} \quad r^2\dot{\varphi} = L$$

the condition $v_\alpha v^\alpha = \delta_1$ $[\delta_1 = 0$ light
$= -1$ Timelike$]$

gives $-f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\varphi}^2 = \delta_1$

Putting everything together,

$$\dot{r}^2 = f\left[\frac{E^2}{f} - \frac{L^2}{r^2} + \delta_1\right] \equiv V$$

a. For $\delta_1 = -1$ (timelike)

$V' = 0 \Rightarrow L^2 = \frac{ME^2 r}{f^2}$, plugging back onto V,

$V = 0 \Rightarrow E^2 = \frac{rf^2}{r - 3M}$

- Circular orbits are only possible for $r \geq 3M$
- Circular orbits are stable for $r \geq 6M$

b. For $\delta = 0$ (light)

$V = 0 \Rightarrow L^2 = \frac{r^2}{f} E^2$

thus $\frac{r^2 E^2}{f} = \frac{M E^2 r}{f^2} \Leftrightarrow r = 3M$

Null, circular geodesics are possible only at $r = 3M$. They are unstable, and define the photosphere.

c. For $E = 1$, we get (it's a quadratic in r)

$$\dot{r}^2 = 1 - f\left(\frac{L^2}{r^2} + 1\right) = \frac{2M^2}{r^3}\left(\frac{r^2}{M^2} + \frac{L^2}{M^2} - \frac{1}{2}\frac{L^2}{M^2}\frac{r}{M}\right)$$

$$= \frac{M^2}{r^3}\left(\frac{r}{M} - \frac{L^2/M^2 + \frac{L}{M}\sqrt{\frac{L^2}{M^2} - 16}}{4}\right)\left(\frac{r}{M} - \frac{L^2/M^2 - \frac{L}{M}\sqrt{\frac{L^2}{M^2} - 16}}{4}\right)$$

For $L > 4M$ there is a turning point at $r > 2M$

If we define $b = \frac{L}{E}$ then $b_{crit} = 4M$

We can repeat this calculation for light and get $b_{crit} = 3\sqrt{3}\,M$, and the ~~tur~~ critical turning point is at $r = 3M$, the photosphere

$\therefore$ Light can never reach points at $r < 3M$ and "live to tell"

BHs have an effective cross section $\pi b^2 = 27 M^2$

d. For radial plunges $L=0$ and we have

$$\dot{r}^2 = E^2 + f\delta_1$$

For a particle falling from ~~rest at~~ r_0 with $E=1$, we have

$$\Delta\tau = \frac{\sqrt{2}\,(r_0/M)^{3/2} - 4}{3}$$

However $\frac{dt}{d\tau} = \frac{E}{f}$ and therefore $\Delta t \to \infty$ logarithmically when $r \to 2M$

these objects, vacuum all the way to $r=2M$ were called at the beginning, "frozen stars".

$r=2M$ is an infinite redshift surface, which we call the event horizon, making up the BH spacetime.

The horizon is a one-way membrane, but this doesn't mean the BHs are boring!

- their appearance is dictated by the photosphere
- their response to "kicks" is too.

In vacuum, the field equations [Field_Equations.nb] immediately force g to be time-independent and

$f = g = 1 - \frac{2M}{r}$ this is the Schwarzschild solution

this is, in essence, also the Birkhoff theorem:

A spherically symmetric vacuum solution is necessarily static

Corollary: In vacuum, the ~~only~~ unique spherically symmetric solution is the Schwarzschild geometry

Extension to stationary non-trivial

Let us examine what happens when we kick a black hole. I take a form of matter to be described by a "scalar field",

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{K} - \frac{1}{2} g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - \mu^2 \psi^2 \right)$$

Varying this action (do it!) leads to

$$\nabla_\mu \nabla^\mu \psi = \mu^2 \psi$$ [Note: follows from $\nabla_\mu T^{\mu\nu} = 0$!]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = K \left[\frac{1}{2} \psi^{,\mu} \psi^{,\nu} - \frac{1}{4} g^{\mu\nu} \left(\psi_{,\alpha} \psi^{,\alpha} + \mu^2 \psi^2 \right) \right]$$

I will take $\|\psi\| \ll 1$, such that it backreacts little on the geometry and can be neglected on the r.h.s of the Einstein equation. Then, we have

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0 \qquad (A)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \psi \right) = \mu^2 \psi \qquad (B)$$ [i will set $\mu = 0$ for now]

A solution to (A), we just saw, is the Schwarzschild geometry. To solve B, we decompose

$$\cancel{\psi = \sum_{lm} \frac{\Phi(r,t)}{r} Y_{lm}(\theta,\varphi)} \qquad \psi = \sum_{lm} \frac{\Phi(t,r)}{r} Y_{lm}(\theta,\varphi)$$

since the spherical harmonics form a complete set. We arrive at [Notebook: scattering_scalars]

$$\frac{\partial^2}{\partial r_*^2} \Phi - \frac{\partial^2}{\partial t^2} \Phi - V \Phi = 0, \text{ where}$$

$$\frac{\partial r_*}{\partial r} = \frac{1}{f} = \frac{1}{1 - \frac{2M}{r}} \quad \& \quad V = f \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right]$$

i. Are there non-trivial static solutions of this equation, describing a BH surrounded by a scalar field?

In staticity, we have

$$\left[\left(1-\frac{2M}{r}\right)\Phi'\right]' - \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)\Phi = 0$$

Multiplying by Φ^* and integrating from $r=2M$ to ∞,

$$\int_{2M}^{\infty} \left[\left(1-\frac{2M}{r}\right)\Phi'\right]'\Phi^* - \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)|\Phi|^2 = 0$$

$$\Rightarrow \left[\left(1-\frac{2M}{r}\right)\Phi'\Phi^*\right]_{2M}^{\infty} - \int_{2M}^{\infty}\left(1-\frac{2M}{r}\right)|\Phi'|^2 - \int_{2M}^{\infty}\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)|\Phi|^2 = 0$$

the first term drops out for regular solutions.
the two remaining are negative-definite.
the only solution is $\Phi = 0$

this result is part of the no-hair & uniqueness results of GR: Kerr-Newman is the only regular, asymptotic solution of electro-vacuum GR.

ii. How does the field disappear? To do that, we need to solve the time-dependent equation.
This is done in the Scattering-scalars.nb

We first define the Laplace transform

$$\tilde{\Phi} = \int_0^\infty dt \; \Phi(\tau, r)\, e^{-s\tau}$$

(I use $s = -i\omega$ in the routine)

then we get the following ODE

$$\frac{d^2\tilde{\Phi}}{dr_*^2} + (\omega^2 - V)\tilde{\Phi} = i\omega\Phi(\tau=0) - \frac{\partial\Phi}{\partial\tau}(\tau=0) \equiv I(r_*)$$

As initial conditions, i use $\Phi(\tau=0) = 0$

$$\frac{\partial\Phi}{\partial\tau}(\tau=0) = f\, e^{-(r_*-r_0)^2/2\sigma^2}$$

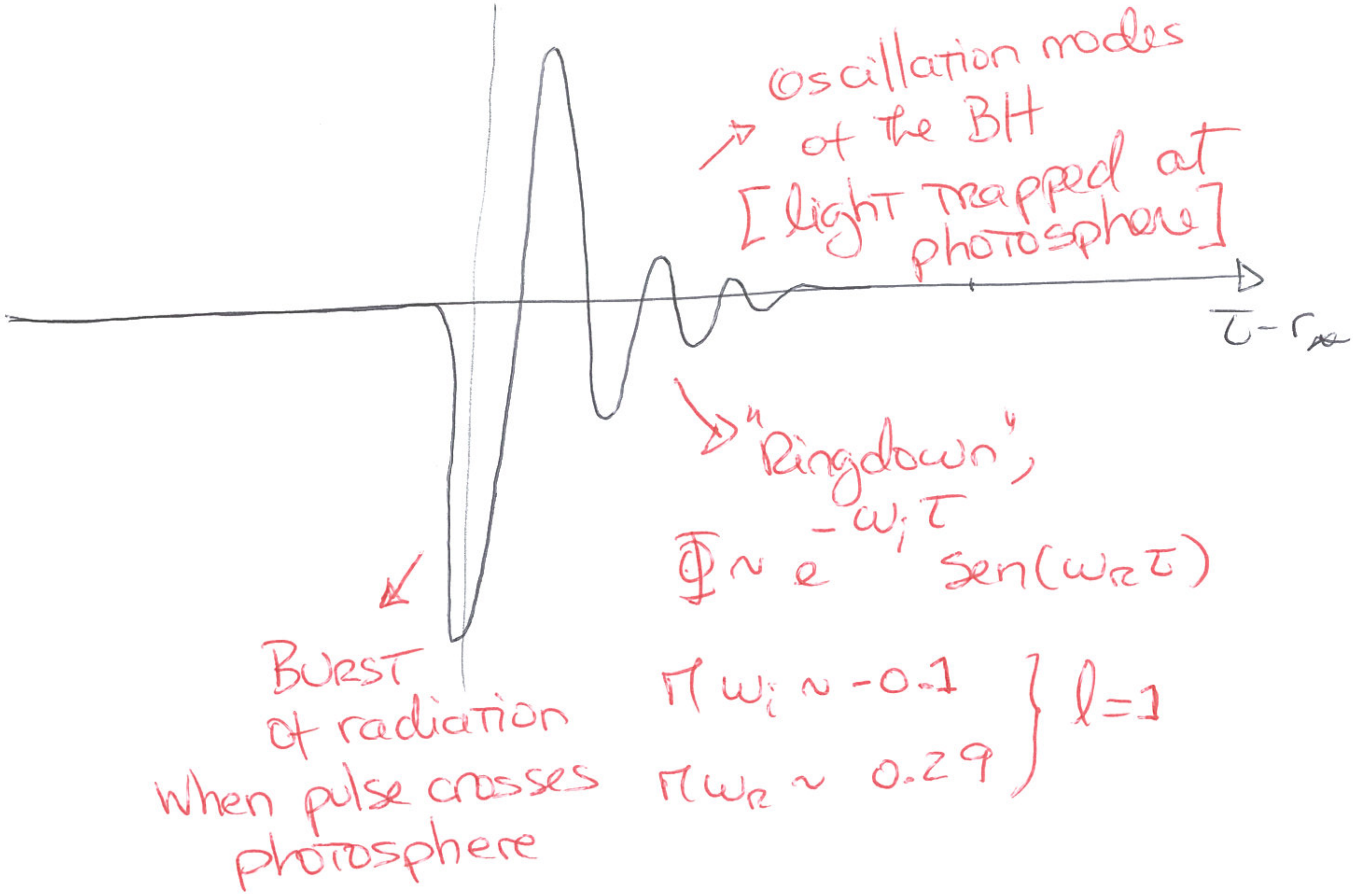
the ~~sol~~ solution satisfying appropriate BCs is, at infinity,

$$\Phi(\omega, r) = e^{i\omega r}\int_{-\infty}^{\infty} \frac{I(r_*)\,\psi_L}{2i\omega A_{in}}\, dr_*$$

when ψ_L is the function which solves ~~the~~ the homogeneous version of the ODE and behaves has

$$\psi_L = \begin{cases} e^{-i\omega r_*} & x \to -\infty \\ A_{in} e^{-i\omega r_*} + A_{out} e^{+i\omega r_*}, & x \to +\infty \end{cases}$$

The solution looks like this



This pattern is common to almost all processes involving plunges into BHs. LIGO's detected signal shares the main characteristics

We can repeat the procedure for Maxwell fields or gravitational fluctuations... but we need to know how to expand them in a convenient basis of angular functions on the sphere.

Take vectors: produce set of complete vector harmonics by applying a set of operators on scalar harmonics, e.g.,

$$L Y_{lm} = \left(0,\ i r \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{lm},\ -i r \sin\theta \frac{\partial}{\partial\theta} Y_{lm}\right)$$

$$\vec{e}_r Y_{lm} = \left(Y_{lm},\ 0,\ 0\right)$$

In 4D language, one can show that

$$A_\mu(t,r,\theta,\phi) = \sum_{lm} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{a_{lm}}{\sin\theta}\partial_\phi Y_{lm} \\ -a_{lm}\sin\theta\,\partial_\theta Y_{lm} \end{bmatrix}}_{(-1)^{l+1}} + \underbrace{\begin{bmatrix} f_{lm}\, Y_{lm} \\ h_{lm}\, Y_{lm} \\ k_{lm}\, Y_{lm} \\ k_{lm}\, Y_{lm} \end{bmatrix}}_{(-1)^{l}}$$

is a convenient expansion. $(-1)^{l+1}$ $(-1)^l$ when

$$(\theta,\phi) \to (\pi-\theta, \pi+\phi)$$

$$\nabla_\mu F^{\mu\nu} = 0 \Rightarrow$$

$$\frac{\partial^2}{\partial r_*^2}\psi - \frac{\partial^2}{\partial t^2}\psi - V\psi = 0, \quad V = f\left[\frac{l(l+1)}{r^2}\right]$$

$$\psi = \begin{cases} a_{lm} \\ r^2(\partial_t h_{lm} - f'_{lm}) \end{cases}$$

Previous analysis for static show $l=0$ is exception!

$f_{lm} = \frac{Q}{r}$ ... charged black hole!

How do things move around black holes?

~~the Lagrangian for a point particle is basically~~

Around Schwarzschild,

$$\dot{r}^2 = f\left[\frac{E^2}{f} - \frac{L^2}{r^2} + \delta_1\right] \equiv V$$

$f = 1 - \frac{2M}{r}$ and $\delta_1 = -1$ (timelike) or $\delta_1 = 0$ (light)

a. For $\delta_1 = -1$

$V' = 0 \Rightarrow L^2 = \frac{M E^2 r}{f}$, plugging back onto V

$V = 0 \Rightarrow E^2 = \frac{r f^2}{r - 3M}$

- Circular orbits are only possible for $r \geq 3M$
- Circular orbits are stable for $r \geq 6M$

b. For $\delta = 0$ (light) $\quad V = 0 \Rightarrow L^2 = \frac{r^2 E^2}{f}$

$\Rightarrow V' = 0 \Rightarrow r = 3M$ unstable!

Defines photosphere

- Connection with ringdown
- Cross section is dictated by light ring:

$b_{crit} = 3\sqrt{3}\,M$

BH cross section $= \pi b^2 = 27\pi M^2$

Stars in GR

Stars in GR are built as they were in Newtonian theory, but all forms of energy gravitate and enter the stress-tensor $T_{\mu\nu}$. For perfect fluids,

$$T^{\mu}{}_{\nu} = \mathrm{diag}\left[-\rho, P, P, P\right]$$

$$T = 3P - \rho$$

Let's take the metric as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{1}{1-\frac{2m(r)}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Einstein equations [Notebook] give the Oppenheimer-Volkoff equations

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{\rho m}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}\left(1+\frac{P}{\rho}\right)^{-1}$$

Need EOS

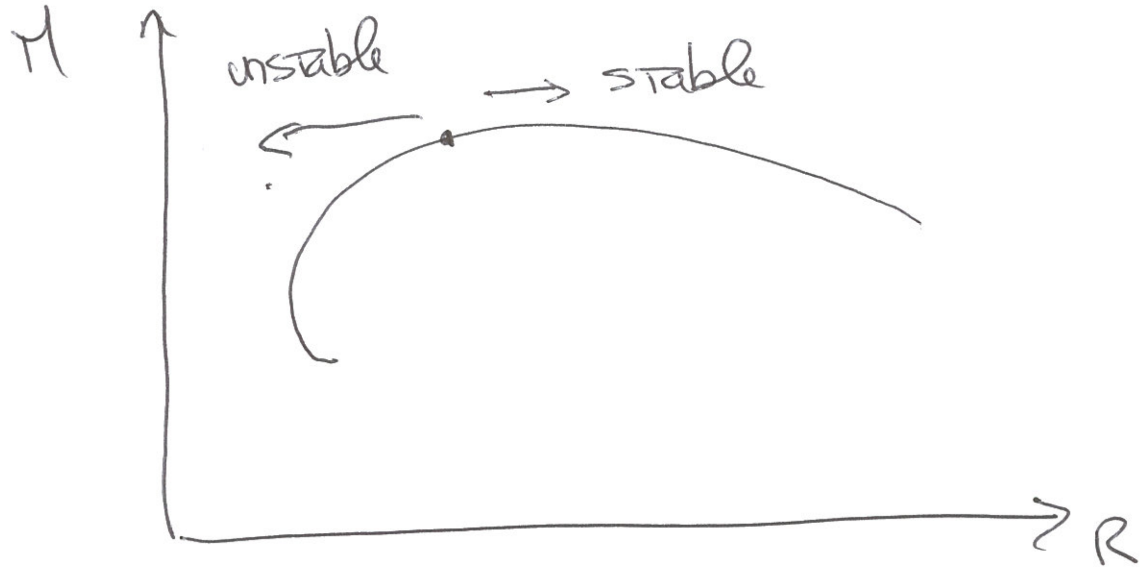
• For constant density stars

$$\frac{P}{\rho} = \frac{\left(1 - 2Mr^2/R^3\right)^{1/2} - \left(1 - 2M/R\right)^{1/2}}{3\left(1 - \frac{2M}{R}\right)^{1/2} - \left(1 - 2Mr^2/R^3\right)^{1/2}}$$

$$e^{\Phi} = \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}$$

$$P_c < \infty \Rightarrow \frac{2M}{R} < \frac{8}{9}$$

stability proceeds as before... for politropics, usually find [~~at~~ notebook]



Beyond GR: this is for you!

- Something is out there [DM, DE]
- We expect something to be out there [mostly QM effects]
- We have no idea where they will show up

Lovelock's theorem does not kill alternatives: Add matter, break diff. invariance, etc.

Popular example: Add scalar fields.

Ex: $S = \int d^4x \sqrt{-g} \left(\frac{R}{K} - \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi^*_{,\nu} - \mu^2 \varphi \varphi^* \right)$

φ complex

Non-trivial physics: Boson stars!

$$\nabla_\mu \nabla^\mu \varphi = \mu^2 \varphi$$

$$\frac{1}{K}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = \frac{1}{4}\left[\varphi^{*,\mu}\varphi^{,\nu} + \varphi^{,\mu}\varphi^{*,\nu}\right] - \frac{1}{2} g^{\mu\nu}\left[\frac{1}{2}\varphi^*_{,\lambda}\varphi^{,\lambda} + \frac{\mu^2}{2}\varphi\varphi^*\right]$$

Time-independent solutions do not exist
[show this, by expanding close to r=0]

But $\varphi \sim e^{-i\omega t} \cdot Z(r)$ leads to possible regular solutions

[Notebook]

Ex 2: $S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}\xi R\varphi^2\right)$

the field equations now give

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\Box\varphi = R\xi\varphi$$

I will not attempt to solve the full set of equations, but i start by

a. Pointing out that $\varphi = 0$ is a solution. It describes the world (we think) we know. Let's take a star which solves $G_{\mu\nu} = 8\pi T_{\mu\nu}$, with $T = P - 3p$

$$\Rightarrow R = 8\pi(p - 3P)$$

The term $R\xi = 8\pi\xi(p-3P)$ acts like μ^2, a mass-squared term. For very compact stars it can become negative ... to see the problem look at homogeneous scalars:

$$\Box\varphi = -\mu^2\varphi \Leftrightarrow \frac{\partial^2}{\partial t^2}\varphi = \mu^2\varphi \Rightarrow \varphi = Ae^{\mu t} + Be^{-\mu t}$$

... it blows up, it's unstable! So a zero scalar might not be the "spontaneous solution" *

b. then, let's take a star in GR, say with constant density, and see if a zero scalar is unstable or not.

* think of a vertically-placed pencil on a table
It is a solution, but unstable

I do this in the notebook, where you can see that there are unstable solutions. In fact, the threshold can be understood analytically:

Take $\varphi = \frac{\Phi(r)}{r}$ $[\omega \sim 0]$ then we get in the Newtonian regime

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{\Phi}{r}\right)\right] = 8\pi\zeta(\rho - 3P)\frac{\Phi}{r}$$

$$\Leftrightarrow \Phi'' - 8\pi\zeta(\rho-3P)\Phi = 0 \Leftrightarrow \Phi'' - \eta\,\Phi = 0$$

$$\Leftrightarrow \Phi = A\cos\sqrt{\eta}\,i r + B\sin\sqrt{\eta}\,i r \quad , \text{inside}$$

$$= 0 \quad \text{outside}$$

Regularity gives $B=0$ (Φ' has to be zero, to have vanishing energy density at the origin)

thus, we have the continuity requirement at the surface

$$i\sqrt{\eta}\,R = \pm\frac{\pi}{2} \qquad \text{for } P\sim 0,\ \rho = \frac{M}{\frac{4}{3}\pi R^3},$$

$$\Leftrightarrow 8\pi\zeta \cdot \frac{M}{\frac{4}{3}\pi R^3}R^2 = -\frac{\pi^2}{4} \Leftrightarrow \boxed{24\zeta M = -\pi^2 R}$$