THE STRONG CP PROBLEM

Refs : Weinberg vol 2 chapters 22 & 23

Kolb & Turner

Peccei & Ritz "Axions as probes of NP"

hep-ph / 0504231

RPP. Raffelt : "Stars as labs for fundamental physics"

We start with
$$\psi' = \left(\exp[i\alpha\gamma_5 T]\right)\psi$$
$$\bar{\psi}' = \bar{\psi}\left(\exp[i\alpha\gamma_5 T^+]\right)$$

$$\Rightarrow \delta\mathcal{L} = \partial_\mu\alpha\, J_5^\mu - \frac{\alpha g^2}{8\pi^2} F^a_{\mu\nu}\tilde{F}^{b\,\mu\nu}\, \mathrm{tr}(T^a T^b T)$$

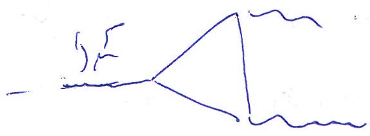
For Weyl spinors $\to \frac{1}{16\pi^2}$

If α is constant, $\Rightarrow \partial_\mu\alpha = 0 \Rightarrow$ global symmetries.

Otherwise can write $\partial_\mu\alpha\, J_5^\mu = -\alpha\,\partial_\mu J_5^\mu$

$\Rightarrow$ varying over α gives the anomaly eq^n.

$\langle \quad \rangle = 0$



Suppose we consider ~~$\psi' = e^{i\alpha\gamma_5 T}\psi$ etc~~

Weyl spinors with constant α.

$\bar{\psi} \to \bar{\psi} e^{-i\alpha T^+} = \bar{\psi} e^{-i\alpha}$

Then $\psi \to e^{i\alpha T}\psi = Z\psi$

$$\alpha T = -i \log Z$$

$$\Rightarrow \delta\mathcal{L} = \frac{i g^2}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{b\,\mu\nu}\, \mathrm{tr}(T^a T^b \log Z)$$

Suppose we diagonalise the quark mass matrix

$$\mathcal{L} \supset -(M_U^{\dagger})_{ij}\, u_{Li} u_{Rj} - (M_D^{\dagger})_{ij}\, d_{Li} d_{Rj}$$

$u_L = T_{UL}^{\dagger} u_L'$ & etc, $u_R = T_{UR}^{\dagger} u_R'$

so then $\Rightarrow T_{UL}^{\dagger} M_U T_{UR} = D^{M_U}$

$$\Rightarrow \delta\mathcal{L} = \frac{ig^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \left[\log(T_{UL}^{*}) + \log T_{UR}^{*} + \log(T_{DL}^{*}) + \log(T_{DR}^{*})\right]$$

$\Rightarrow \operatorname{tr}\log T_{UL}^{\dagger} + \operatorname{tr}\log T_{UR}$

$= \log\det(T_{UL}^{\dagger} T_{UR})$

$= \log\det M_U - \log\det m_U$

$= \arg\det M_U$

$\Rightarrow M_U = T_U^{\dagger} M_U T_{UR}$

now $\log\det|T_U^{\dagger} M_U T_{UR}|$

$= \log\det|T_{UR} T_U^{\dagger}| + \log\det M_U$

$= \log\det m_U$

∴ the action picks up

$$\mathcal{L} \supset -(\arg\det M)\frac{g_3^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$$

Note that this violates CP!

How large is it? NA easy to relate to V_{CKM}.

Would expect generally $O(1)$ phases. Even if not

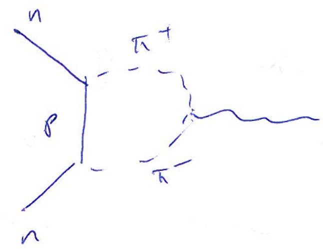
δ_{13} in CKM is ~ 1.2 rad

So any effect on detections is non-perturbative only.

②

Effect of $-\bar{\theta}\frac{g^2}{32\pi^2} F\tilde{F}$:

Neutron EDM.



Calculation is complicated in chiral perturbation theory.

Obtain $d_n \approx |\bar{\theta}| \frac{e m_\pi^2}{m_N^3} \approx 10^{-16} |\theta|\, e\,cm$.

Observed ~~value~~ limit is $|d_n| < 10^{-26}\, e\, cm$

$\Rightarrow |\theta| < 10^{-10}$!!

This represents a strange fine-tuning with no anthropic explanation.

Solution proposed by Peccei & Quinn: make θ dynamic.

Suppose we break a global symmetry $\phi \to e^{i\alpha}\phi$ via ~~a~~ spontaneously, $\langle\phi\rangle = f$. If there are fermions which also transform under this symmetry we can write

: $\phi = f e^{\frac{ia}{f}}$ ~~in~~ (heavy longitudinal cpt integrated out)

Remove the phase via $\varphi \to e^{-\frac{ia}{f}}\varphi$, $\psi \to e^{-iq_\psi a}\psi$

so then

$$\delta\mathcal{L} = -\frac{a}{f}\left[\partial_\mu j_5^\mu\right] + \frac{g^2}{8\pi^2} F^a_{\mu\nu}\tilde{F}^{b\mu\nu}\, \mathrm{tr}\left(T^a T^b Q_a\right)$$

So we shift the theta-angle to

$$\bar{\theta} = \theta + a\, \sum C_2(\psi)\, q_\psi$$

At the QCD phase transition, we expect $\langle F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\rangle$ to have a vev, generating a potential for $(\theta + \frac{a}{f})^2 \Rightarrow$ stabilising it at

$\bar{\theta} = \theta - \frac{a}{f} = 0$!

~~Alone~~ More explicitly:

In fact, ~~the~~ ~~presence of~~ this is not quite true. The $F_{\mu\nu}\tilde{F}^{\mu\nu}$ vev actually gives a mass to the η': ~~ie~~ Since the charm, bottom & top quarks are heavier than Λ_{QCD}, the u, d, s strongly feel QCD effects. QCD spontaneously breaks the SU(3) isospin; but the symmetry is anomalous under QCD, and the anomaly gives mass to the $\eta_8 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \approx \eta'$ (η, η' mix, $\eta_1 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$)

Instead, the axion potential can be seen from the ~~chiral~~ interaction with pions.

Let us treat the strange quark as heavy; then let us suppose we have

$$\mathcal{L} \supset \left(\theta + \frac{a}{f}\right)\frac{g_s^2}{32\pi^2} F\tilde{F} + \frac{a}{f} C_{a\gamma\gamma}\frac{e^2}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$- m\bar{\psi}\psi + \frac{C_{aff}}{2 f_a}\partial_\mu a\left(\bar{f}\gamma^\mu\gamma_5 f\right)$$

Then we can ~~shift~~ remove the QCD anomaly via

$$u \to e^{-i\alpha_u a \gamma_5} u, \quad d \to e^{-i\alpha_d a \gamma_5} d$$

s.t. $\alpha_u + \alpha_d = -\frac{1}{2}\left(\theta + \frac{a}{f}\right)$

so put $\alpha_{u,d} = -\frac{1}{2} C_{u,d}\left(\theta + \frac{a}{f}\right)$ for now.

now shift the axion: $\frac{\bar{a}}{f} \equiv \theta + \frac{a}{f}$.

The ~~mass~~ derivative terms ~~are not invariant:~~ derivative terms pick up an extra shift

$\delta\mathcal{L} = $ ~~$\bar{q}\,\alpha\,\partial a \ldots$~~

$$\frac{C_{aff}}{2f} \to \frac{C_{aff}}{2f} - \frac{\alpha_u}{2 f_a} \quad \text{etc.}$$

Then we consider what happens at low energy: 3

$\begin{pmatrix} u \\ d \end{pmatrix}$ ~~$= \exp($~~ ← fermion ver, dimension $3/2$

$u = \exp\left(\frac{i}{\sqrt{2}}\frac{\pi^0}{F_\pi}\gamma_5\right)v$ $\quad \bar{u} \to v^\dagger \exp\left(\frac{-i}{\sqrt{2}}\frac{\pi^0}{F_\pi}\gamma_5\right)\gamma^0 = \bar{v}\exp\left(\frac{i}{\sqrt{2}}\frac{\pi^0}{F_\pi}\gamma_5\right)$ ...

$\bar{d} = d = \exp\left(-\frac{i}{\sqrt{2}}\frac{\pi^0}{F_\pi}\gamma_5\right)v$

Want $i\bar{u}\gamma^\mu\partial_\mu u \to \frac{1}{4}\partial_\mu\pi^0\partial^\mu\pi^0 + \ldots$

$\therefore \quad \bar{u}\gamma^\mu = -\gamma_5\frac{f_\pi\bar{v}\partial^\mu\pi^0}{2|v|^2} + \ldots$

$\bar{u}\gamma^\mu\gamma_5 u \to -\frac{F_\pi}{2}\partial^\mu\pi^0 + \ldots$

$\bar{d}\gamma^\mu\gamma_5 d \to \frac{f_\pi}{2}\partial^\mu\pi^0 + \ldots$

$\bar{u}\gamma^\mu u \to 0 + \mathcal{O}(\pi^2)$

and $-m_u\bar{u}u \to -\frac{m_\pi^2}{4}(\pi^0)^2$ without the anom.

So we have $-m_u\bar{u}\exp(-2\alpha_u\gamma_5)u$ → include extra shift of π in θ

$\to +m_u\bar{v}\exp\left(\frac{i}{2}\frac{\pi^0}{F_\pi}\gamma_5\right)\exp(-2i\alpha_u\gamma_5)\exp\left(\frac{i}{2}\frac{\pi^0}{F_\pi}\gamma_5\right)v$

$= +m_u\bar{v}\exp\left(\left(\frac{i\pi^0}{F_\pi} - 2i\alpha_u\right)\gamma_5\right)v$

$= +m_u\bar{v}v\cos\left(\frac{\pi^0}{F_\pi} - 2\alpha_u\right)$

$-m_d\bar{d}(\quad)d = +m_d\bar{v}v\cos\left(-\frac{\pi^0}{F_\pi} - 2\alpha_d\right)$

Also have kinetic mixing terms

$\mathcal{L} \supset \frac{F_\pi}{4f}\left(C_{add} - c_d - C_{a\bar{u}u} \pm c_u\right)\partial_\mu\pi\,\partial^\mu a$

~~Or choose $c_u = c_d$ to cancel this~~

Either: choose $c_u = c_d$ to cancel this, or

Can make a shift $a \to a + \frac{i}{2}(\quad)$

→ affects ~~so~~ everything at $\mathcal{O}\left(\frac{f_\pi}{f}\right) \ll 1$.

$m_u(\bar{u}_L u_R + \bar{u}_R u_L)$
$u_L \to e^{i\alpha}u_L$, $u_R \to e^{-i\alpha}u_R$ ... is chiral ...)

~~Mass matrix~~

expand the coins:

$$\mathcal{L} \supset |v|^2 (m_u + m_d)$$

$$- \frac{|v|^2}{2}\left[m_u \left(\frac{\pi^0}{f_\pi} - 2\alpha_u\right)^2 + m_d \left(\frac{\pi^0}{f_\pi} + 2\alpha_d\right)^2\right] + \dots$$

$$\to -\frac{|v|^2}{2}\left[(c_d m_d - c_u m_u) \frac{a\,\pi^0}{f f_\pi} + (m_u + m_d) \frac{\pi^{0\,2}}{f_\pi^2} + (m_u c_u^2 + m_d c_d^2) \frac{a^2}{f_a^2}\right]$$

$\Rightarrow$ choose $c_d m_d = c_u m_u = (1 - c_d) m_u \Rightarrow c_d (m_u + m_d) = m_u \Rightarrow c_d = \frac{m_u}{m_u + m_d}$

$$\Rightarrow m_\pi^2 = |v|^2 \frac{(m_u + m_d)}{f_\pi^2} \Rightarrow \boxed{|v|^2 = \frac{m_\pi^2 f_\pi^2}{m_u + m_d}} \quad c_u = \frac{m_d}{m_u + m_d}$$

$$m_a^2 = \frac{|v|^2 (m_u c_u^2 + m_d c_d^2)}{f_a^2} = \frac{m_\pi^2 f_\pi^2}{m_u + m_d} \frac{}{f_a^2} \quad (\quad)$$

$$= \frac{m_\pi^2 f_\pi^2}{f_a^2} \cdot \frac{m_u m_d^2 + m_d m_u^2}{(m_u + m_d)^3}$$

put $z = \frac{m_u}{m_d} \sim 0.56$

$$\therefore m_{ud} \; \frac{z + z^2}{(1+z)^3} = \frac{z}{(1+z)^2}$$

$$\therefore m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2} \cdot \frac{z}{(1+z)^2}.$$

Then ~~the~~ recall the coupling to photons is (4)

$$+ \frac{a}{f} C_{a\gamma\gamma} \frac{e^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} .$$

But need $u \to \exp(i\alpha_u \gamma_5)\, u$, $d \to \exp(-i\alpha_d \gamma_5)$ under em:

$\Delta C_{a\gamma\gamma}$ $\delta \mathcal{L} = + \frac{g^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \, \mathrm{tr}\left(Q^2 \cdot (\alpha_u + \alpha_d)\right)$

$$\Rightarrow \Delta \mathcal{L}_{\alpha} = 4 \times 3 \times \left(\frac{4}{9} \alpha_u + \frac{1}{9} \alpha_d\right)$$

$$\Delta C_{a\gamma\gamma} = - \frac{4}{3} \times \left(\frac{4}{3} \cdot c_u + \frac{1}{3} c_d \right) \times \frac{1}{2}$$

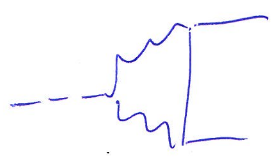
$$= - \frac{4}{3} \cdot \frac{1}{2} \left(\frac{4}{3} \frac{m_d}{m_u + m_d} + \frac{1}{3} \frac{m_u}{m_u + m_d} \right)$$

$$= - \frac{4}{3} \cdot \frac{1}{2} \left(\frac{4}{3} \frac{1}{1+z} + \frac{1}{3} \frac{z}{1+z} \right)$$

$$= - \frac{2}{3} \frac{4+z}{1+z}$$

so can't escape a photon coupling!

The other very important coupling is to electrons, which are not affected by this ... but can have a loop



$$\Rightarrow \quad \Delta C_{iee} = \frac{3\alpha^2}{4\pi^2} C_{a\gamma\gamma} \log \frac{f}{m_e}$$

Nucleon couplings can be extracted from $\langle N | \bar{q} \gamma_\mu \gamma_5 q | N \rangle$ etc.

Because the axion mix with pions

E with strength $\frac{f_\pi}{2f} \left(C_{add} - C_d - C_{auu} + C_u \right)$

we can copiously produce axion if f is small;

pions from e.g. beam dump experiment $\sigma_{pp} \sim 40\,\mathrm{mb}$,

Past bound of $f_a > 3\,\mathrm{TeV}$.

Astrophysical bounds are _much_ stronger:

red giant cooling: $\frac{f}{C_\gamma} > 10^7\,\mathrm{GeV}$

SN 1987 A: $\frac{f}{C_\gamma} \gtrsim 10^8\,\mathrm{GeV}$ (γ's)

white dwarf cooling: $f \gtrsim 2 \times 10^9\,\mathrm{GeV}$ if $C_{ae} \sim 1$

G117−B15A white dwarf: $\frac{f}{C_{ae}} \gtrsim 3.9 \times 10^9\,\mathrm{GeV}$

KSVZ

~~Kim~~ Shifman, Vainshtein, Zakharov

~~[illegible]~~ $V = [illegible]\ y\,(\varphi\,\psi_R \psi_L + h.c.)$

$$- m^2 |\varphi|^2 + \lambda |\varphi|^4$$

for a heavy quark charged under QCD

~~This has two $U(1)$s under $\varphi \to e^{i\alpha}\varphi$, $\psi_L \to e^{i\alpha}\psi$ or ψ_R...~~

or indeed $\psi_L \to e^{-i\alpha}\psi$

$\psi_R \to e^{-i\alpha}\psi$

$\varphi \to e^{2i\alpha}$

~~[illegible]~~

The other $U(1) \subset SU(3)$ where $\psi_L \to e^{i\alpha}\psi_L$, $\psi_R \to e^{-i\alpha}\psi_R$!!

This model has no couplings to SM quarks, so no EM anomaly

So $C_{a\gamma\gamma}$ is $-\frac{2}{3}\frac{4+z}{1+z}$

$C_{aee} = \frac{3\alpha^2}{4\pi^2} C_{a\gamma\gamma} \log\frac{f}{m_e}$

DFSZ

Dine Fischler Srednicki

(6)

~~SUSY~~ $\mathcal{L}_Y = G_u \bar{q}_L \phi_u u_R + G_d \bar{q}_L \phi_d d_R + h.c.$

$\phi_u \to \exp(i\alpha X_u)\phi_u \qquad \phi_d \to \exp(i\alpha X_d)\phi_d$

$\phi \to \exp(i\alpha X_\phi)\phi$

$X_u + X_d = -2X_\phi = 1$

$\langle \phi_u \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} f_u \\ 0 \end{pmatrix} \qquad \langle \phi_d \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ f_d \end{pmatrix}$

$\langle \phi \rangle = \frac{1}{\sqrt{2}} f_\phi \gg \sqrt{f_u^2 + f_d^2} \equiv f$

$$V = \lambda_u(|\phi_u|^2 - v_u^2)^2 + \lambda_d(|\phi_d|^2 - v_d^2)^2 + \lambda(|\phi|^2 - v^2)^2 + (a|\phi_u|^2 + b|\phi_d|^2)|\phi|^2 + c(\phi_u \cdot \phi_d \phi^2 + h.c.) + d|\phi_u \cdot \phi_d|^2 + e|\phi_u^* \phi_d|^2$$

So SUSY model with

$$W = \frac{S^2}{\Lambda} H_u \cdot H_d + \lambda_S X(ST - f^2)$$

Here we 'solve' the μ-problem too!

Say $u_R \to -X_u \qquad$ & $e_R \to -X_d$

$d_R \to -X_d$

then C_{QED} : $(3 \times 3 \times \frac{4}{9} X_u + 3 \times 3 \times \frac{1}{9} X_d + 3 \times X_d)$

$\uparrow$ $[\phi_{KSI}]$ $= 4(X_u + X_d)$

$= 4$

But SU(3) anomaly is $\frac{1}{2} \times (3X_u + 3X_d) = \frac{3}{2}$

$\therefore E/N = 8/3$.

With DFSZ, ~~we have Yukawa couplings so couplings~~

~~to electrons:~~

$$\cancel{m_e \bar{\Psi}\Psi \leftarrow y_e \bar{E}_L e_R H_d}$$

$$\cancel{\downarrow}$$

$$\cancel{e^{-i a X_d}}$$

~~me~~ m_e

Yukawa invariant.

so couplings to electrons from kinetic term

$$i \bar{\Psi} \not{\partial} \Psi \to \frac{1}{2} \partial_\mu a \; \bar{e} \gamma^\mu \gamma_5 e \; \frac{X_d}{f}$$

~~Note~~ Note this gives $C_{aee} = 2X_d$

(Sometimes define f to include QCD anomaly)

In original paper, took $X_u = \dfrac{v_d^2}{f^2}$, $X_d = \dfrac{v_u^2}{f^2}$

Cosmology of axions & ALPs

Now will talk more generally: if we allow one axion, why only one? UV physics (ST) ~~predicts~~ suggests an 'axiverse' of PNGBs, with logarithmically spread masses. There would not get one from QCD → only one linear combination in mixes with the pion & become massive!

Have seen that $\frac{f}{C_{i\gamma}} \gtrsim 10^8$ ~~GeV~~ GeV

$\Rightarrow \quad m_a \lesssim 0.1$ eV for QCD.

~~Inter~~ Interestingly, there are suggestions of ~ massless ALPs

- White dwarf cooling is anomalously fast, suggests $\frac{f_a}{C_{ie}} \sim 10^9$ GeV

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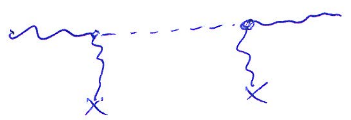
- Anomalous transparency of the universe



$\Rightarrow \quad \sim \frac{f}{C_{i\gamma}} \sim 10^8$ GeV

- A ~~the~~ background density of ALPs remaining from the CMB could ~~explain~~ a large (multi-sigma) excess of soft ~~X~~-rays from the Coma cluster (1312.3947, 1305.3603)

- Can look for ALPs with $\frac{f_a}{C_{i\gamma}} \sim 10^5$ GeV (currently) in LSW exps. Will improve to $\frac{f}{C_{i\gamma}} \sim 10^8$ GeV in future!

- Searching for ALPs from the Sun $\Rightarrow$ IAXO $\frac{f}{C_{i\gamma}} \sim 10^8$ GeV ~~CT~~

- In Direct detection experiments $\Rightarrow$ Horizontal bound.

LATEST bounds see exp. review of J. Redondo

e.g. 1501.01639

For heavier ALPs & mass e.g. ~ GeV, can search for in meson decays or at beam dump expts!

AXION DARK MATTER

Main production is misalignment.

In expanding universe, axion /ALP is a scalar field which oscillates

$$\ddot{\varphi} + 3H\dot{\varphi} + m_\varphi^2 \varphi = 0$$

Note that $H \simeq \frac{1.66\, g_*^{1/2} T^2}{M_P}$

At high T, φ is approximately massless.

When the mass becomes greater than the expansion rate, $m_a \gtrsim 3H$, the free axion starts to oscillate with frequency m_a.

At this point, the energy trapped in the field will decrease, but not the number density:

$$n_{axion}(\text{initial}) = \frac{\rho(T_1)}{m(T_1)} \simeq \frac{1}{2} m_a(T_1)\, \overline{\theta}_i^2 f^2$$

$$n_{axion}(\text{today}) = \frac{s_{today}}{s_{initial}} \times n_i$$

Do this first for an ALP: the mass is constant.

Then $m \sim \frac{1.66\, g_*^{1/2} T^2}{M_P}$

$$s = \frac{2\pi}{45} g_{*S} T^3$$

$$\frac{s_{today}}{\rho_c/h^2} \sim 0.28\ eV^{-1}$$

$$\Rightarrow \frac{\Omega h^2}{0.112} \sim 1.4 \times \left(\frac{m_i}{eV}\right)^2 \left(\frac{f_a}{10^{11}\,GeV}\right)^2 \left(\frac{\langle\overline{\theta}_i^2\rangle}{\pi^2/3}\right)$$

for the QCD axion, for $T \gg$ GeV, $m_a = 0$

for $T \ll$ very small, $m_a \simeq \frac{m_\pi f_\pi}{f_a} \sqrt{\frac{z}{(1+z)^2}}$

~~the behaviour~~ 10^{-37} GeV $< H <$ 10^{-23} GeV is important.

$\Downarrow$

$T \sim$ eV

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we find that $T_1 \sim \left(\frac{m}{10^{-5}\,\text{eV}}\right)^{1/6}$ GeV

$\sim 10^{\frac{14}{6}} \left(\frac{m}{\text{GeV}}\right)^{1/6}$ GeV

$\Rightarrow \Omega h^2 \sim \frac{90}{\rho_c/h^2} \times \left(\frac{m_\pi f_\pi}{f_a}\right) \times \frac{1}{\frac{2\pi}{45} \times g_{*s} T_1^3} \times \frac{1}{2} \bar{\theta}^2 f^2 \times 3 \frac{1.66\, g_*^{1/2} T_1^2}{M_p}$

$\sim 4 \times \frac{m_\pi f_\pi}{\frac{2\pi}{45} g_{*s}} \frac{3}{2} \bar{\theta}^2 \frac{f}{M_p T_1} \times 1.66\, g_*^{1/2}$

$T_1 \sim 10^{\frac{14}{6}} \left(\frac{m}{\text{GeV}}\right)^{1/6}$

$\sim 10^{\frac{14}{6}} \left(\frac{m_\pi f_\pi}{f_a}\right)^{1/6}$

so $\frac{\Omega h^2}{0.12} \sim 6.3 \times \left(\frac{f_a}{10^{12}\,\text{GeV}}\right)^{7/6} \times \left(\frac{\langle\theta\rangle}{\pi}\right)^2$

so unless we tune θ, we usually say that

10^9 GeV $< f_a <$ 10^{12} GeV.

However, this can be relaxed

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To understand the ~~mass~~ temperature dependence of axion mass, need results of

GROSS, PISARSKI & YAFFE.

Instanton density is

$$n(\rho,T) = \frac{C_N}{\rho^5}\left(\frac{1}{\alpha_s}\right)^{2N_c}\left(\prod_{i=1}^{N_f}(\xi\rho m_i)\right)$$
$$\times \exp\left[-\frac{8\pi^2}{g^2} + \frac{1}{3}\lambda^2(2N+N_f) + 12A(\lambda)\left[1+\tfrac{1}{6}(N-N_f)\right]\right]$$

Here $\lambda \equiv \pi\rho T$.

The energy density is then $\int d\rho\, n(\rho,T)$

$\xi = 1.33876$, $\quad C_N = \dfrac{(0.260156)\,\xi^{-(N-2)}}{(N-1)!(N-2)!}$

$$\frac{8\pi^2}{g^2} = \frac{8\pi^2}{g_0} - \frac{1}{3}(11N - 2N_f)\log\rho\bar{\mu} + \dots$$

set $\dfrac{8\pi^2}{g_0} = 0$ at $\bar{\mu} = \Lambda$, $\rho = 1$ $= \dfrac{2}{\alpha_s} = -\dfrac{1}{3}(11N - 2N_f)\log\rho\Lambda$

So now for three quark flavours, & 3 colours, energy density is

$$V = C_N\,\xi^3 \int d\rho\,\frac{1}{\rho^5}\left(\frac{1}{\alpha_s}\right)^6 \rho^3\,(m_u m_d m_s)$$
$$\times \exp(9\log\rho\Lambda)\exp(f(\lambda))$$

where $f(\lambda) \simeq -\xi 3\lambda^2 + \dots$

put $\rho = \frac{\lambda}{\pi T} \Rightarrow V = C_N\,\xi^3\,\dfrac{(m_u m_d m_s)\Lambda^9}{(\pi T)^8}\displaystyle\int_0^\infty d\lambda\left(\frac{1}{\alpha_s}\right)^6 \lambda^7 \exp(-3\lambda^2 + \dots)$

so $m_a^2 f_a^2 \sim T^{-8} \to m_a \sim \dfrac{(m_u m_d m_s\Lambda^9)^{1/2}}{f_a T^4} \cdot \alpha_s^{-3} \times \dots$

so put $m_a(T_1) = H$

$$= \frac{1.66\, g_*^{1/2}\, T^2}{M_p}$$

$$\Rightarrow \quad T^6 = \frac{M_p}{1.66\, g_*^{1/2}} \frac{(m_u m_d m_s \Lambda^9)^{1/2}}{f_a} \times \left(\log \frac{\Lambda}{\pi T}\right)^6 \times \text{const.}$$

n.b.

$$\int_0^\infty d\lambda (\log e\lambda)^6 \lambda^7 e^{-3\lambda^2}$$

$$\to \int_0^\infty d\lambda \left(\ln\frac{\Lambda}{\pi T} + \ln\lambda\right)^6 \lambda^7 e^{-3\lambda^2}$$

$$\approx \left(\log\frac{\Lambda}{\pi T}\right)^6 \int_0^\infty d\lambda (\log\lambda)^6 \lambda^7 e^{-3\lambda^2}$$

since $\frac{d}{d\lambda}\left(\lambda^7 e^{-3\lambda^2}\right)$ is maxim at $\lambda = \sqrt{7/6}$ & $\log\sqrt{7/6}$ is small...

~~But the energy density~~ is

is ~~Thus~~

$$T^6 = \frac{M_p}{1.66\, g_*^{1/2}} \left(\frac{m_\pi f_\pi}{f_a}\right) \frac{(m_u m_d m_s \Lambda^9)^{1/2}}{m_\pi f_\pi} \times \ldots$$

$$T_1 = (m_a)^{1/6} \times \left[M_p \frac{(m_u m_d m_s \Lambda^9)^{1/2}}{m_\pi f_\pi} \right]^{1/6} \times (\ldots)^{1/6} \times \frac{1}{(\alpha_s(\pi T))^{1/2}}$$

$$\approx (m_a)^{1/6}$$