An ultra-short course on General Relativity

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Astroparticules
et Cosmologie

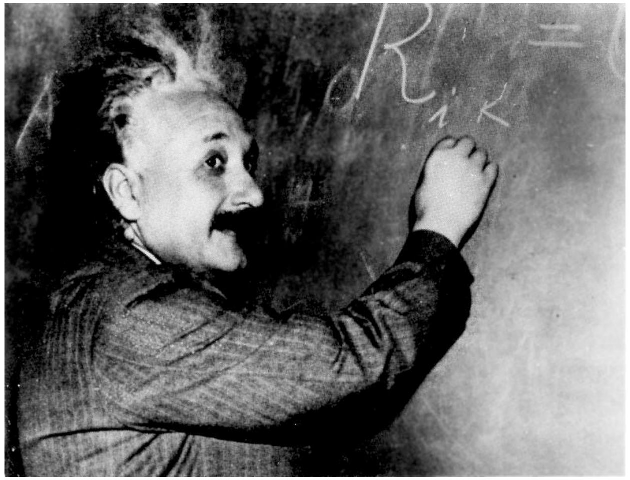
Outline

1. Introduction
2. Geometric tools
3. Einstein’s equations
4. Main physical applications
   - Compact objects
   - Gravitational waves
   - Cosmology

Introduction

General relativity (1915)

- How to extend special relativity to describe gravitation ?



- Gravitation: geometrical deformation of Minkowski spacetime

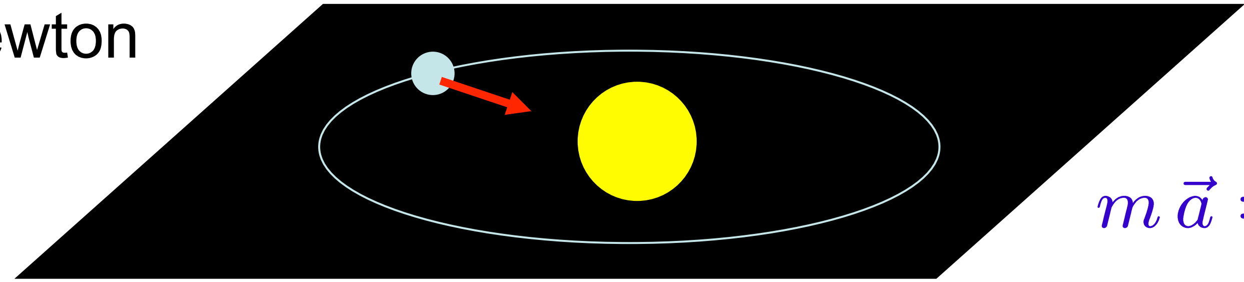
- The spacetime geometry depends on the matter content

- Einstein equations extend Poisson equation $\Delta U = 4\pi G \; \rho_m$

Curvature of spacetime ⟷ Distribution of matter

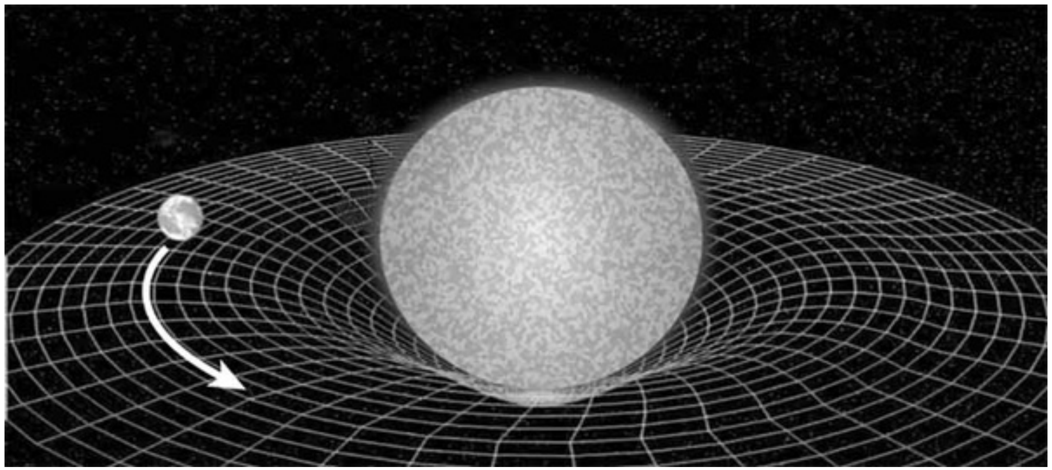
Newton vs Einstein

- Newton



$$m\,\vec{a} = -m\vec{\nabla}\phi_g$$

- Einstein



Free motion in a curved spacetime

$$m\,\vec{a}_{\text{eff}} = 0$$

$$\vec{a}_{\text{eff}} = \vec{a} + \vec{\nabla}\phi_g$$

Geometrical tools

Geometry & metric

- Fundamental object: **metric**

$$d\ell^2 = g_{ij}\, dx^i\, dx^j \qquad\qquad ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

- Measuring distances: $$d(A,B) = \int_A^B d\ell$$

- Change of coordinates: $x^i \longrightarrow \tilde{x}^k$

$$d\ell^2 = g_{ij}\, dx^i\, dx^j = g_{ij}\, \frac{\partial x^i}{\partial \tilde{x}^k}\, \frac{\partial x^j}{\partial \tilde{x}^l}\, d\tilde{x}^k\, d\tilde{x}^l \equiv \tilde{g}_{kl}\, d\tilde{x}^k d\tilde{x}^l$$

Geometry & metric

- **Metric as a scalar product**

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{ij}\, u^i\, v^j$$

- Vectors can be seen as (directional) derivative operators

For any function f, $\boldsymbol{u}(f) := u^i \dfrac{\partial f}{\partial x^i}$

Hence the notation $\boldsymbol{u} = u^i \dfrac{\partial}{\partial x^i}$

- Change of coordinates

$$\boldsymbol{u}(f) = u^i \frac{\partial f}{\partial x^i} = \tilde{u}^k \frac{\partial f}{\partial \tilde{x}^k} \qquad \tilde{u}^k = \frac{\partial \tilde{x}^k}{\partial x^i} u^i$$

Tensors

- One can also consider arbitrary **tensors**, whose components are labelled by several indices.

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

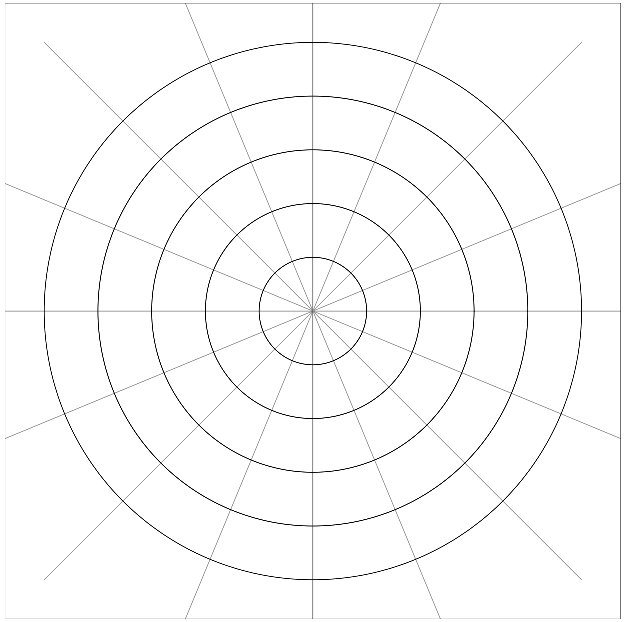
- Mathematically: multilinear maps of vectors or covectors

- One can distinguish
  - Covariant indices: $\tilde{u}_k = \frac{\partial x^i}{\partial \tilde{x}_k} u_i$
  - Contravariant indices: $\tilde{u}^k = \frac{\partial \tilde{x}^k}{\partial x^i} u^i$

Example: polar coordinates

- In 2-dimensional Euclidean geometry

$$x = r\cos\theta, \qquad y = r\sin\theta$$



- Metric

$$d\ell^2 = dx^2 + dy^2 = dr^2 + r^2\, d\theta^2$$

- Vectors

$$\boldsymbol{v} = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta}$$

$$v^x = \cos\theta\, v^r - r\sin\theta\, v^\theta,$$
$$v^y = \sin\theta\, v^r + r\cos\theta\, v^\theta.$$

Covariant derivative

- Derivation of a vector field valid in arbitrary coordinates ?

$\frac{\partial v^i}{\partial x^j}$ are not the components of a tensor.

Example: constant vector field $$\boldsymbol{u} = \frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}$$

- **Covariant derivative**

$$\nabla_i v^k \equiv \partial_i v^k + \Gamma^{k}_{i\,j} v^j \qquad \Gamma^{k}_{i\,j} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}\right)$$

Christoffel symbols

- Polar coordinates:

$$\Gamma^{r}_{\theta\theta} = -r\,, \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \boldsymbol{\nabla u} = 0$$

Covariant derivative

- For arbitrary tensor fields

$$\nabla_i T^{j_1 \dots j_p}{}_{k_1 \dots k_q} = \partial_i T^{j_1 \dots j_p}{}_{k_1 \dots k_q} + \sum_{r=1}^{p} \Gamma_{i\ l}^{\ j_r} T^{j_1 \dots l \dots j_p}{}_{k_1 \dots k_q} - \sum_{s=1}^{q} \Gamma_{i\ k_s}^{\ l} T^{j_1 \dots j_p}{}_{k_1 \dots l \dots k_q}$$

- For the metric tensor

$\nabla_k g_{ij} = 0$ by construction

Motion of a particle

- **Newtonian physics**
  - Parameter: time t
  - Trajectory: $x^i(t)$
  - Velocity $v^i \equiv \frac{dx^i}{dt}$
  - Acceleration $a^i \equiv v^k \nabla_k v^i$

$$a^k \equiv v^i \nabla_i v^k = \frac{dv^k}{dt} + \Gamma^k_{ij}\, v^i v^j$$

e.g. in polar coords

$$a^r = \ddot{r} - r\dot{\theta}^2\,, \quad a^\theta = \ddot{\theta} + \frac{2}{r}\,\dot{r}\,\dot{\theta}$$

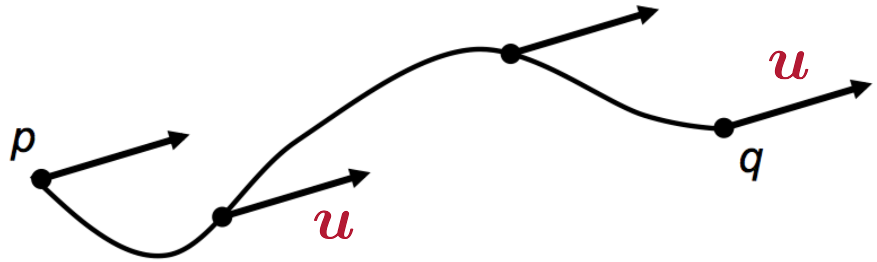
- **Relativistic physics**
  - Parameter: proper time τ
  - Trajectory: $x^\mu(\tau)$
  - Velocity $v^\mu \equiv \frac{dx^\mu}{d\tau}$
  - Acceleration $a^\mu \equiv u^\sigma \nabla_\sigma u^\mu$

Curvature

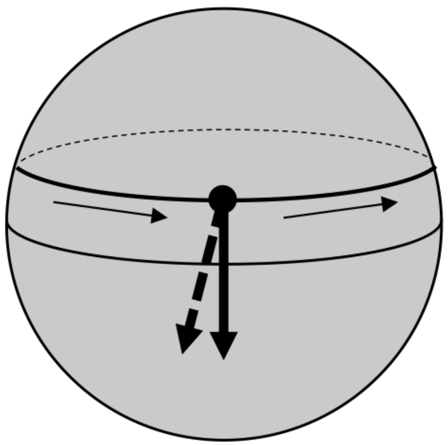
- How to distinguish a curved space(-time) from a flat one ?

- Parallel transport

$$t^i \nabla_i u^k = 0$$

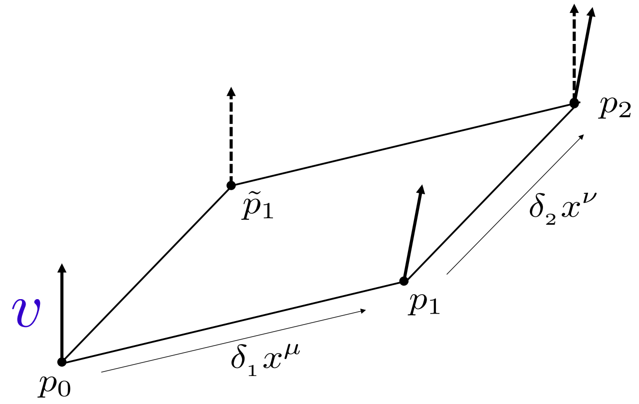


- In flat spaces, vectors are unchanged after parallel transport along a loop. Not always in curved spaces…



Curvature

- Consider an infinitesimal loop and the parallel transport of a vector along two paths.



$$\tilde{v}^\sigma(p_2) - v^\sigma(p_2) = R_{\mu\nu\alpha}{}^\sigma \, v^\alpha \, \delta_1 x^\mu \, \delta_2 x^\nu$$

$$R_{\mu\nu\alpha}{}^\sigma = \partial_\mu \Gamma^\sigma_{\nu\alpha} - \partial_\nu \Gamma^\sigma_{\mu\alpha} + \Gamma^\sigma_{\mu\beta}\, \Gamma^\beta_{\nu\alpha} - \Gamma^\sigma_{\nu\beta}\, \Gamma^\beta_{\mu\alpha}$$

Riemann curvature tensor

- This tensor characterizes the curvature of spacetime (or space). It vanishes for flat spacetimes (or spaces).

Einstein's equations

Gravitation in relativity

- Gravitation due to the **deformation of spacetime**

- Consider a **spherically symmetric** spacetime

$$ds^2 = -\left[1 + 2U(r)\right] dt^2 + \left[1 + 2V(r)\right] dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- Free motion of a particle in this spacetime: $a^\mu \equiv u^\sigma \nabla_\sigma u^\mu$

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

- In the nonrelativistic limit, one gets $\dfrac{dv^i}{dt} \simeq -\partial^i U$

Fictitious force ! (with U as Newtonian potential)

Einstein equations

- How to determine $g_{\mu\nu}$?
- Goal: find the relativistic version of Poisson's equation

$$\Delta U = 4\pi G \, \rho_m$$

- Distribution of matter

$$\rho_m \longrightarrow \rho$$

For a particle, P^μ is conserved, as well as its charge Q

For a distribution of particles

- Current vector $j^\mu = \rho_Q u^\mu$
- Stress-energy-momentum tensor $T^{\mu\nu}$

Einstein equations

- Equations of the form $G_{\mu\nu} \propto T_{\mu\nu}$

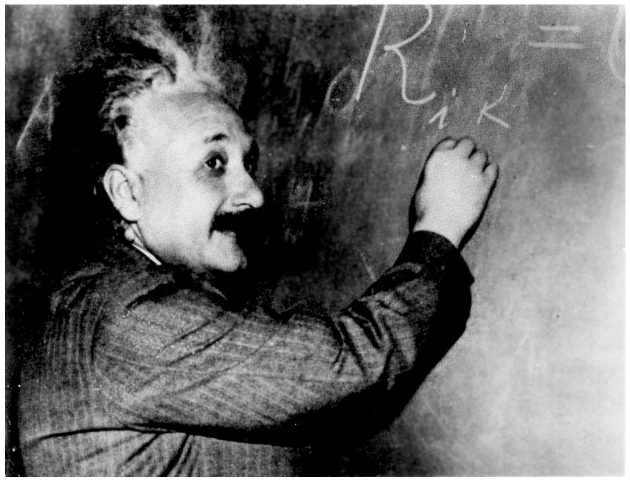
- Gravitational tensor ?
  - must depend on second derivatives of the metric
  - must satisfy $\nabla_\mu G^{\mu\nu} = 0$, since $\nabla_\mu T^{\mu\nu} = 0$

From the Riemann tensor, one can construct the Ricci tensor:

$$R_{\mu\nu} \equiv R_{\lambda\mu\nu}{}^{\lambda}$$

$$\nabla^\mu G_{\mu\nu} \equiv \nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Einstein-Hilbert action

- Einstein's equations can also be derived from a variational principle

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R$$

$$R \equiv g^{\mu\nu}R_{\mu\nu}$$

- Variation of the action

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^4x\,\left[\delta(\sqrt{-g})R + \sqrt{-g}\,\delta(g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\right]$$

Using $\delta g = g\,g^{\rho\sigma}\delta g_{\rho\sigma}$ and $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$

as well as $g^{\mu\nu}\delta R_{\mu\nu} = \nabla^\lambda\left[-g^{\rho\sigma}\nabla_\lambda\delta g_{\rho\sigma} + g^{\rho\sigma}\nabla_\rho\delta g_{\lambda\sigma}\right]$

one gets $\delta S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\frac{1}{2}R\,g^{\rho\sigma} - R^{\rho\sigma}\right]\delta g_{\rho\sigma}$

Einstein-Hilbert action

- One can include matter via the action

$$S_m[\phi, g_{\mu\nu}] = \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

- Defining the energy-momentum tensor as

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_m)}{\delta g_{\mu\nu}}$$

the variation of the total action yields

$$\delta S_{EH} + \delta S_m = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\frac{1}{2} R\, g^{\rho\sigma} - R^{\rho\sigma} + 8\pi G\, T^{\rho\sigma} \right] \delta g_{\rho\sigma}$$

Solution of Einstein's equations

- Einstein's equations are extremely difficult in general
- One often imposes symmetries to solve them
- Spherical symmetry (and staticity) in vacuum

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

- Solving vacuum Einstein's equations, one finds

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Schwarzschild metric (1916)

Geodesics in Schwarzschild

- Metric

$$ds^2 = -\left(1 - 2m/r\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- Free particle with velocity $u^\mu = \{\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}\}$
- Integrals of motion of geodesic eq $\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^{\lambda}_{\mu\,\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$
  - Energy: $E \equiv \left(1 - 2m/r\right) \dot{t}$
  - Angular momentum $L \equiv r^2 \sin^2\theta \, \dot{\phi}$
  - Normalization $-\left(1 - 2m/r\right) \dot{t}^2 + \frac{\dot{r}^2}{1 - 2m/r} + r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right) = -1$

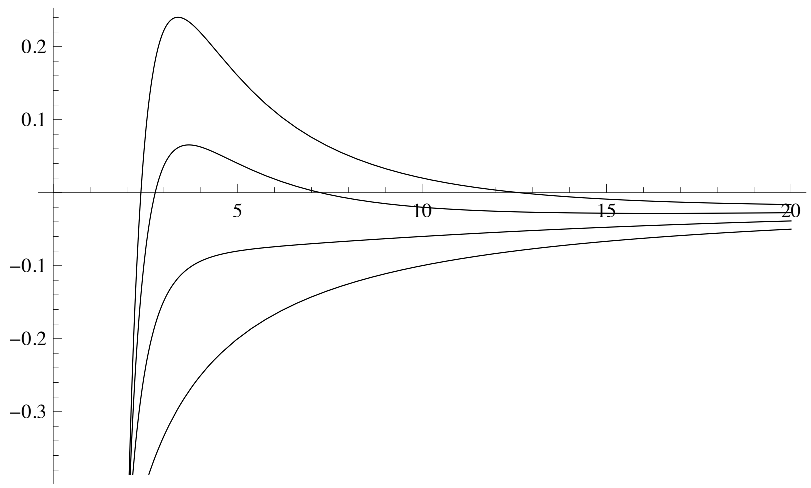
Geodesics in Schwarzschild

- Radial integral of motion

$$\frac{1}{2}\dot{r}^2 + V_{\rm eff}(r) = \frac{E^2 - 1}{2}$$

$$V_{\rm eff}(r) \equiv \underbrace{-\frac{m}{r} + \frac{L^2}{2r^2}}_{} - \frac{mL^2}{r^3}$$

Newtonian
effective potential



- For the Sun, the relativistic correction is proportional to

$$m = \frac{GM_\odot}{c^2} \simeq 1.5\ {\rm km}$$

Trajectories of planets

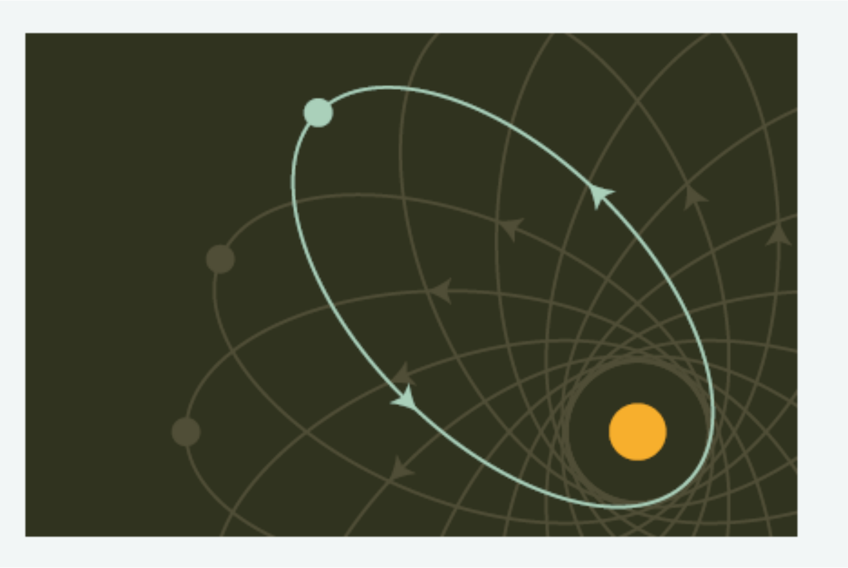
- Newtonian theory: $\frac{1}{r} = \frac{1}{\bar{r}}(1 + e\cos\phi)\,, \quad \bar{r} \equiv \frac{L^2}{\mathcal{M}c^2}$
- General relativity : $\frac{1}{r} \simeq \frac{1}{\bar{r}}\left[1 + e\cos\left(\phi - 3\frac{\mathcal{M}}{\bar{r}}\phi\right)\right]. \qquad e \ll 1$

Advance of the perihelion

$$\cos\left(\phi - 3\frac{\mathcal{M}}{\bar{r}}\phi\right) = 1$$

$$\Rightarrow \quad \phi_0 = 0, \quad \phi_1 = 2\pi + \delta\phi$$

$$\delta\phi \simeq 6\pi\frac{\mathcal{M}}{\bar{r}}$$



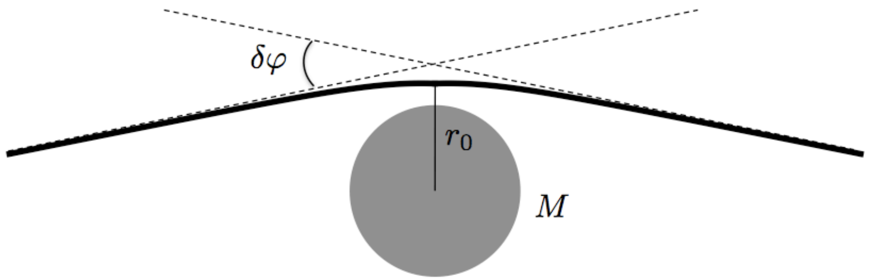
For Mercury, $\bar{r} \simeq 5 \times 10^7$ km $\qquad \Delta\phi \simeq 43''$ per century

Deviation of light

- Light-like trajectory $$g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} = 0$$
- Integral of motion $$\frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} = \frac{E^2}{2}$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - 2m\,u^3 = u_0^2 - 2m\,u_0^3 \qquad (u \equiv 1/r)$$

- Deviation of light



$$\Delta\phi = 2\int_0^{u_0} \frac{du}{\sqrt{u_0^2 - 2mu_0^3 - u^2 + 2mu^3}} \simeq \pi + 4mu_0$$

$$\delta\phi = \frac{4GM}{c^2 r_0}$$

Deviation of light

- Deviation angle

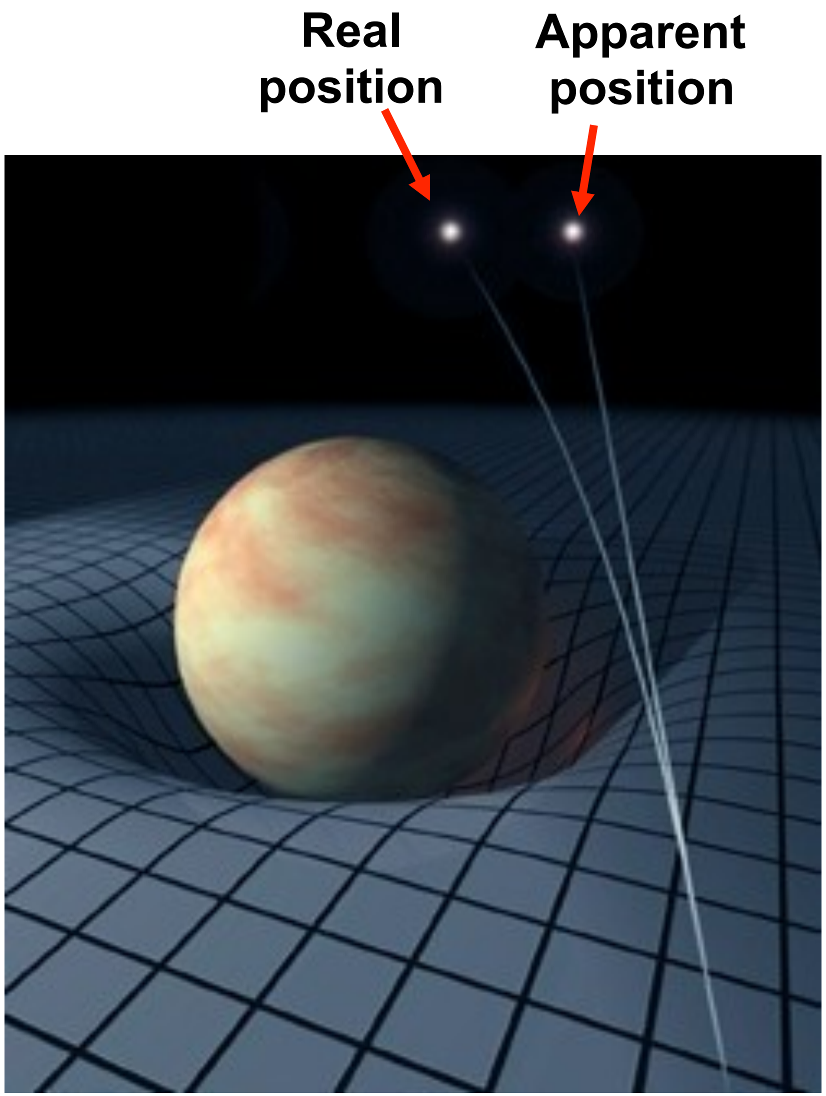
$$\alpha = \frac{4GM}{Rc^2}$$

- Solar eclipse:

$$R_\odot \simeq 700\,000 \text{ km}$$

$$\alpha \simeq 1,75''$$

Observational confirmation in 1919



Relativistic stars

Solving Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

- Spherical symmetry (and staticity)

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

- Outside the star: Schwarzschild metric.

- Inside the star: energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

$$T_{tt} = \rho\, e^{\nu}\,, \qquad T_{rr} = p\, e^{\lambda}\,, \qquad T_{\theta\theta} = p\, r^2\,, \qquad T_{\phi\phi} = p\, r^2 \sin^2\theta$$

TOV equation

- Combining Einstein's equations, one obtains

$$\frac{dp}{dr} = -G\frac{\mathcal{M} + 4\pi r^3\, p}{r^2(1 - 2G\mathcal{M}/r)}(\rho + p)$$

$$\mathcal{M}(r) \equiv 4\pi \int_0^r \rho(\bar{r})\, \bar{r}^2 d\bar{r}$$

Tolman-Oppenheimer-Volkov

- In Newtonian theory, hydrostatic equilibrium $\vec{\nabla} p + \rho\, \vec{\nabla} U = 0$ leads to the relation

$$\frac{dp}{dr} = -\frac{G\mathcal{M}\rho}{r^2}$$

- Existence of maximal masses

Various compact objects

- Compactness parameter $$\Xi \equiv \frac{GM}{Rc^2}$$

- Three types:

  1. White dwarves $\Xi \sim 10^{-3} - 10^{-4}$
  2. Neutron stars $\Xi \sim 0.2$
  3. Black holes $\Xi = 1/2$

Supernovae

- Gravitational collapse of a massive star

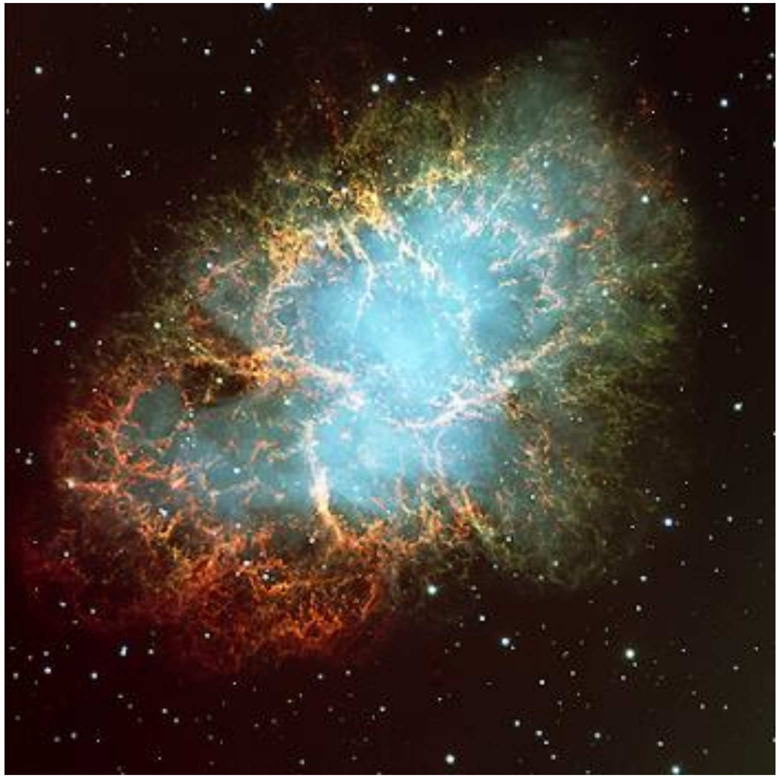
- Birth of a neutron star

$$p + e^- \rightarrow n + \nu$$

- Example: Crab nebula

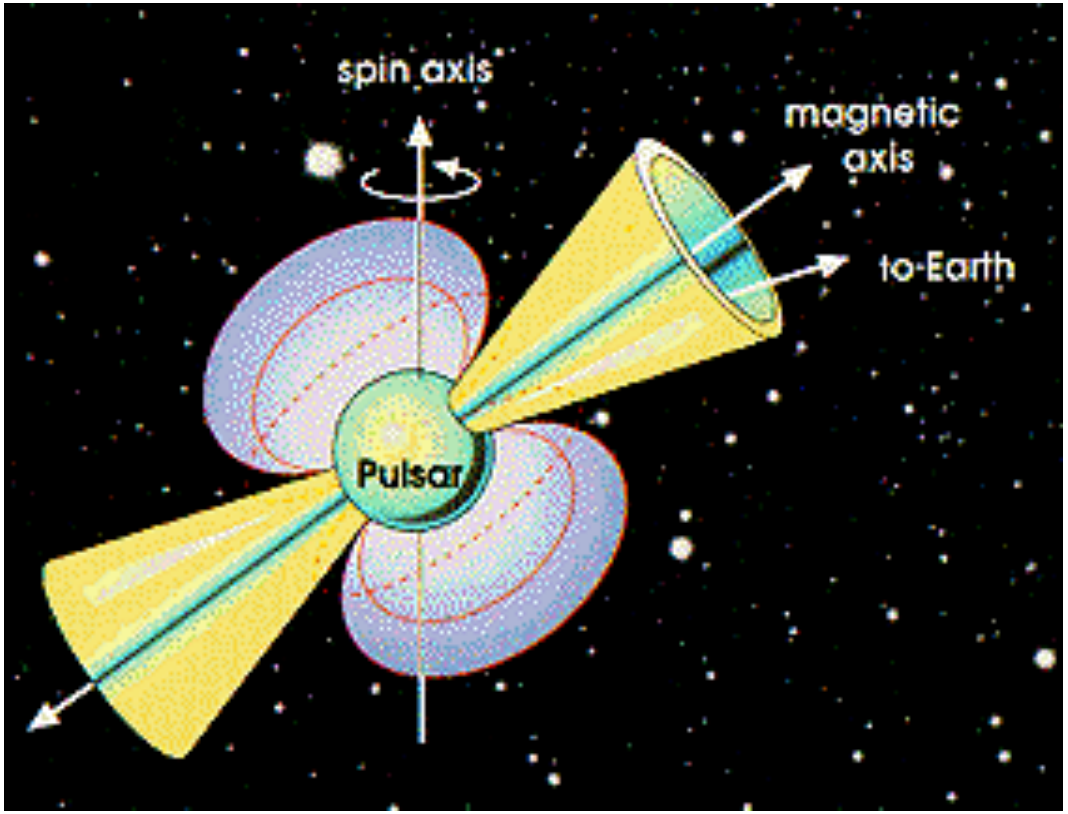
Supernova observed in 1054

Contains a pulsar de of period P=33 ms

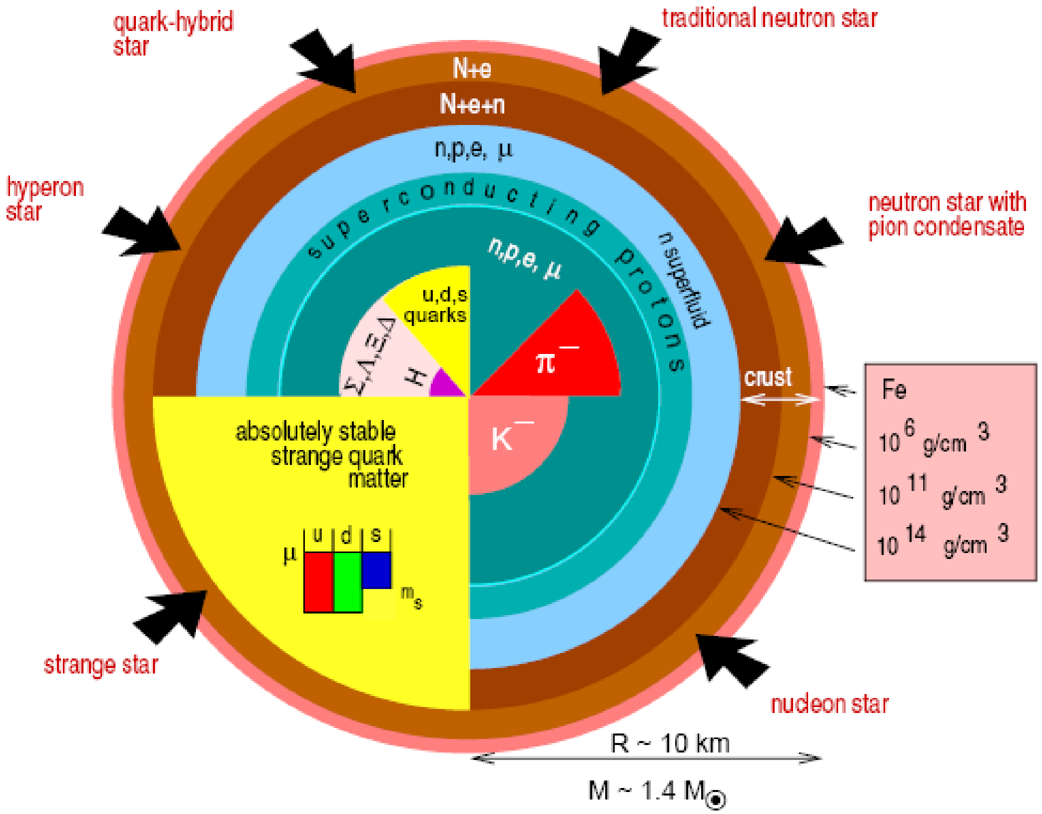


ESO VLT

Pulsars



Interior of neutron stars



Weber(2001)

Black holes

Schwarzschild black holes

- Metric

$$ds^2 = -\left(1 - 2m/r\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- Star collapse: what if the radius reaches $r = 2m$?

- The metric looks singular in $r = 2m$
  - But no curvature singularity
  - Only a **coordinate singularity**, which can be resolved by using new coordinates.

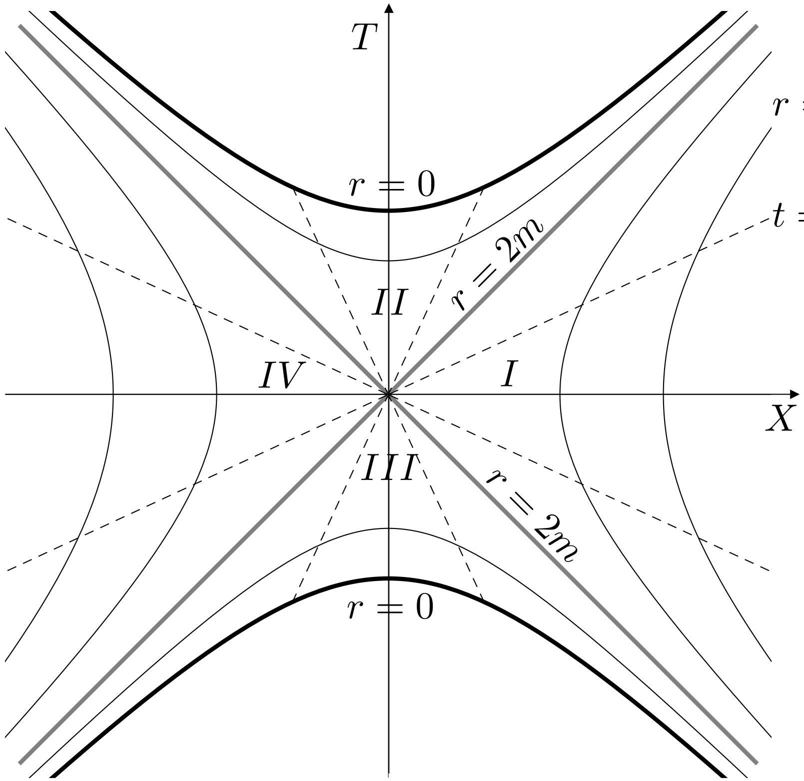
$$r_*(r) \equiv \int \frac{dr}{1 - 2m/r} = r + 2m \ln|r/2m - 1|$$

$$T = [-e^{-(t-r_*)/4m} + e^{(t+r_*)/4m}]/2, \quad X = [e^{-(t-r_*)/4m} + e^{(t+r_*)/4m}]/2,$$

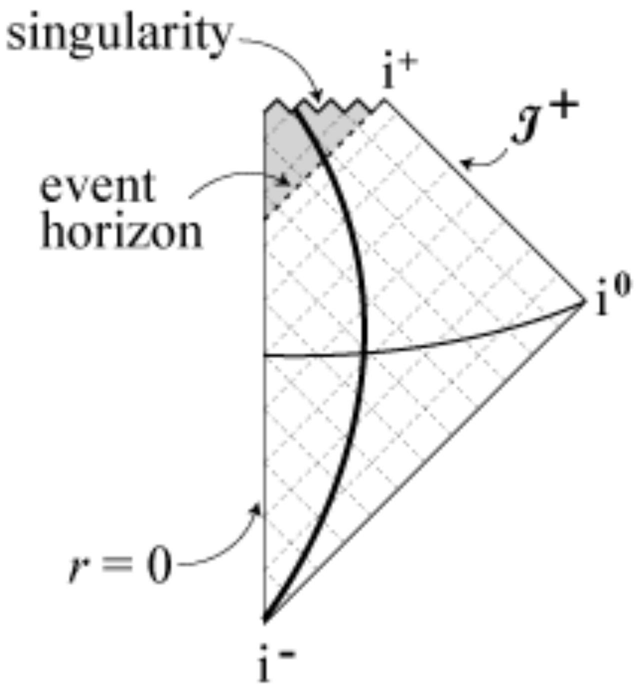
Schwarzschild black holes

- Kruskal coordinates

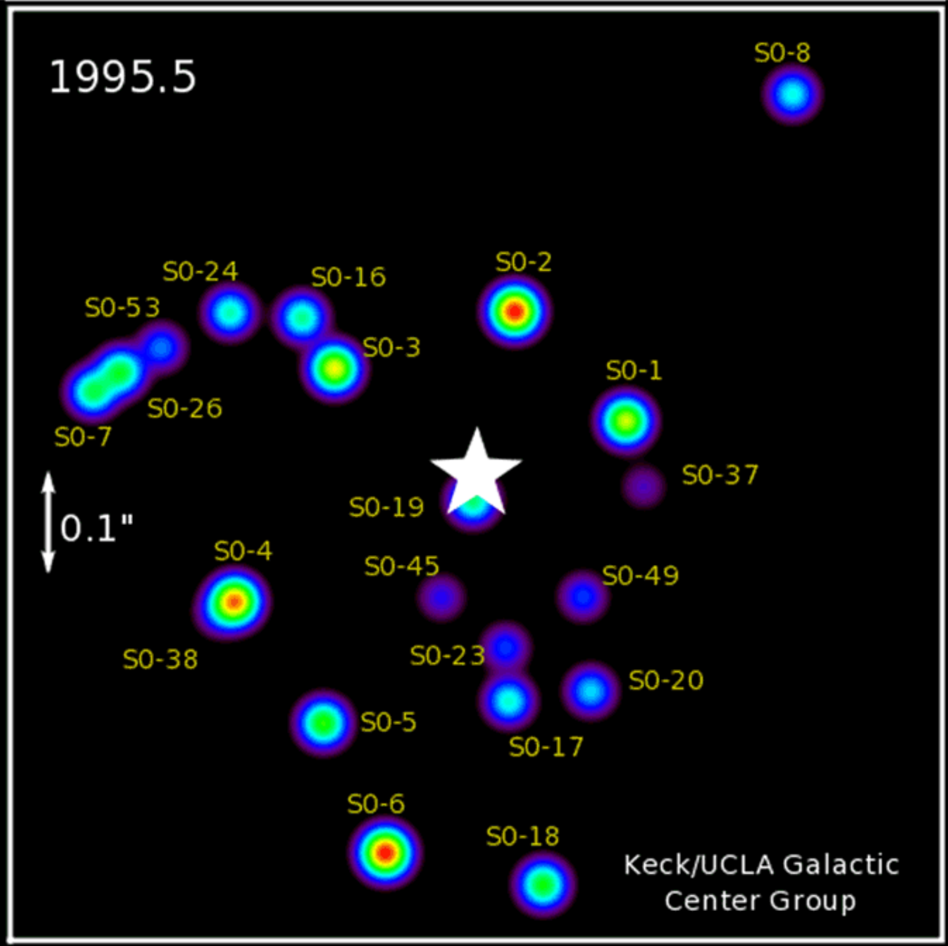
$$ds^2 = 32\frac{m^3}{r}e^{-r/2m}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$



Astrophysical Black Holes



In the core of our galaxy



Gravitational waves

Gravitational waves

- Linearisation of Einstein equations

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

$$\partial_\lambda \partial^\lambda h_{\mu\nu} = S_{\mu\nu}$$

- Analogous to Maxwell equations $\partial_\lambda \partial^\lambda A_\mu = J_\mu$

- Gravitational waves propagate with the speed of light

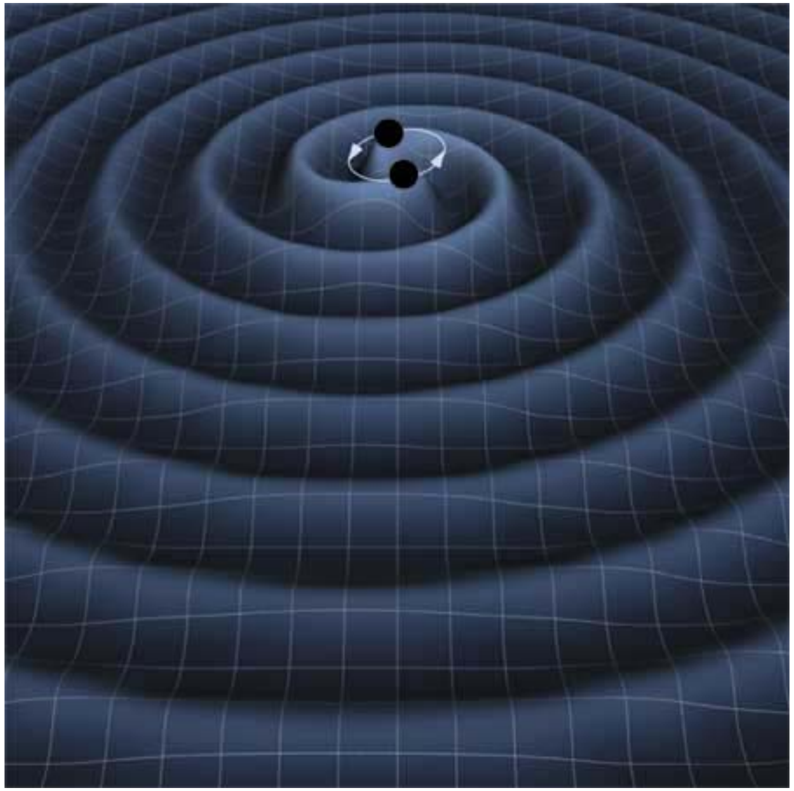
- 2 independent modes

Sources of gravitational waves

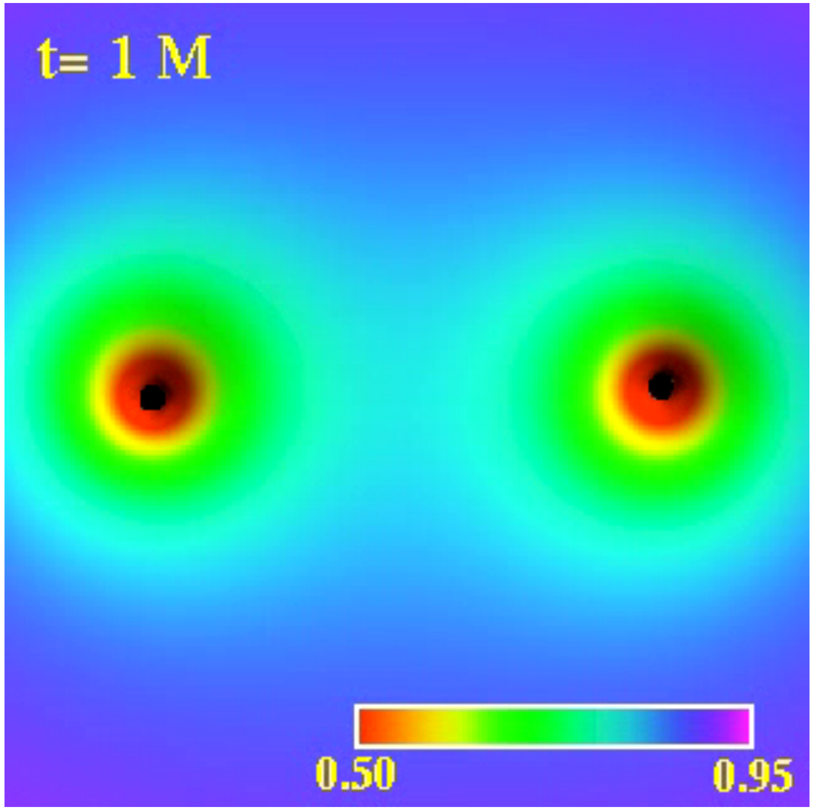
- Coalescence of a binary system (up to 100 Mpc)

- Supernovae (up to 10 Mpc)

- Continuous sources
  (deformation of neutron stars)

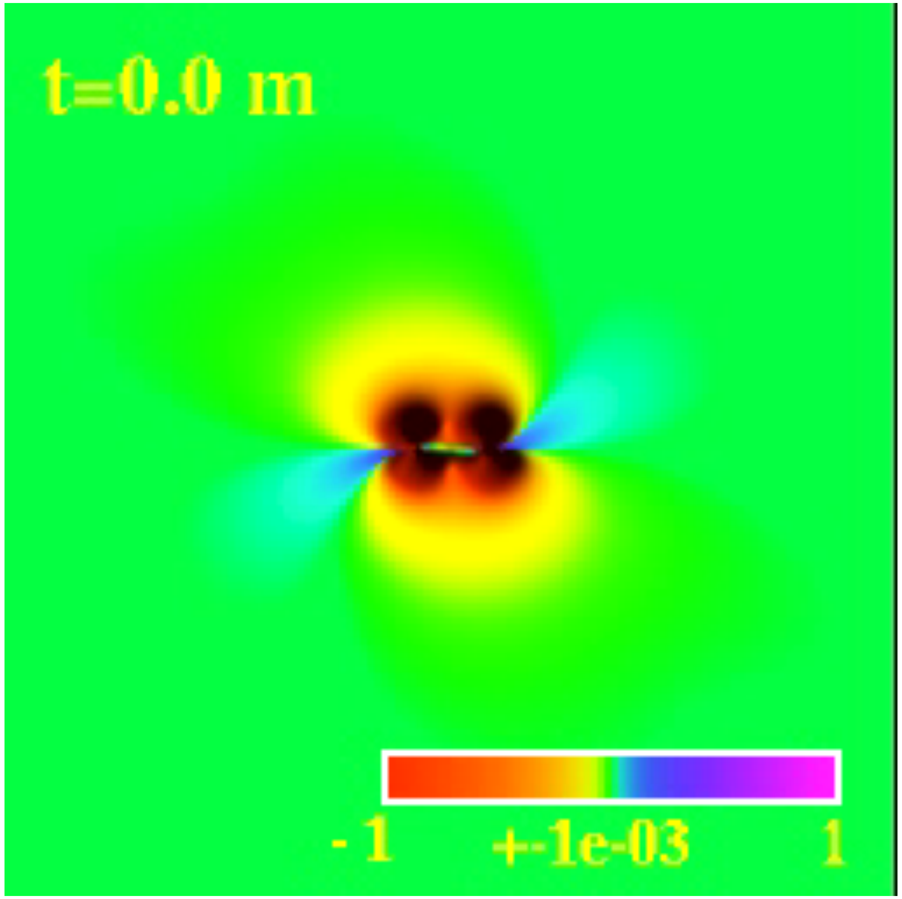


Binary coalescence

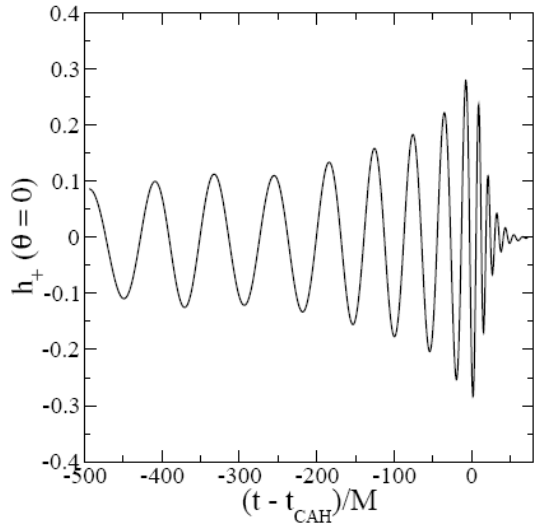


F. Pretorius

Binary coalescence



F. Pretorius



Emission of gravitational waves

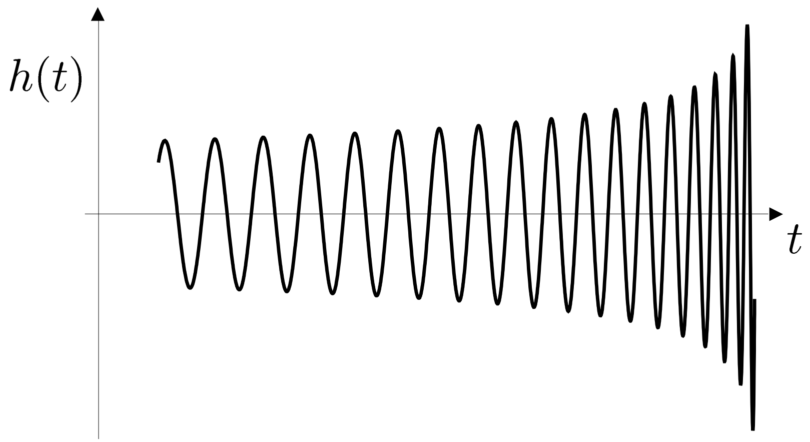
- Energy loss

$$-\left(\frac{dE}{dt}\right)_{\rm grav} = \frac{G}{5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle \qquad Q_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij} I^k_k$$

- Binary system

$$E = -\frac{GM\mu}{2d}$$

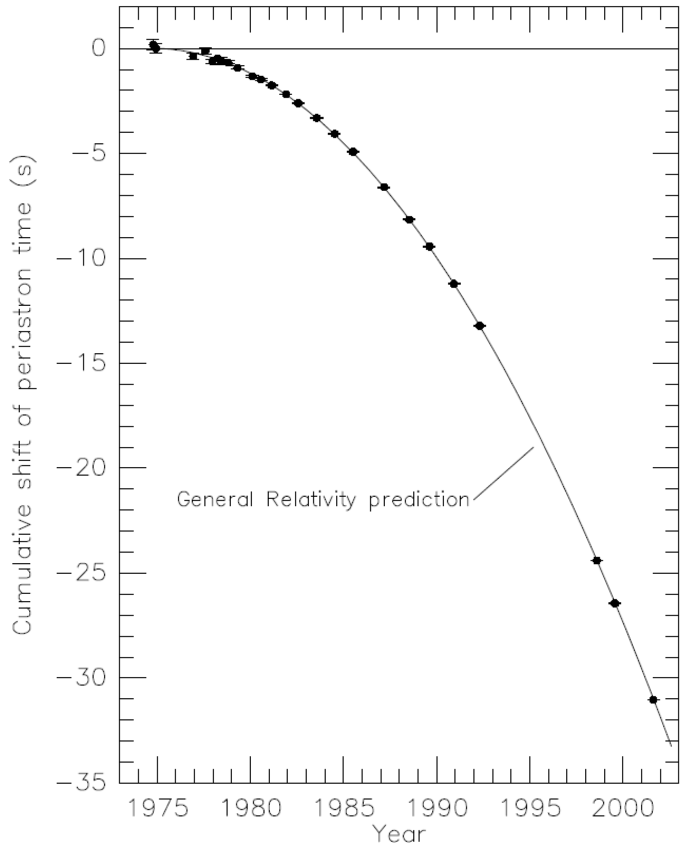
$$\dot{E} = -\frac{32}{5}\frac{G^4\mu^2M^3}{d^5} \sim \frac{c^5}{G}\left(\frac{R_S}{d}\right)^5$$



Indirect detection

Binary pulsar PSR B1913+16 (1974)

Evolution of the orbital period
P=7h45m



[Weisberg & Taylor (2002)]

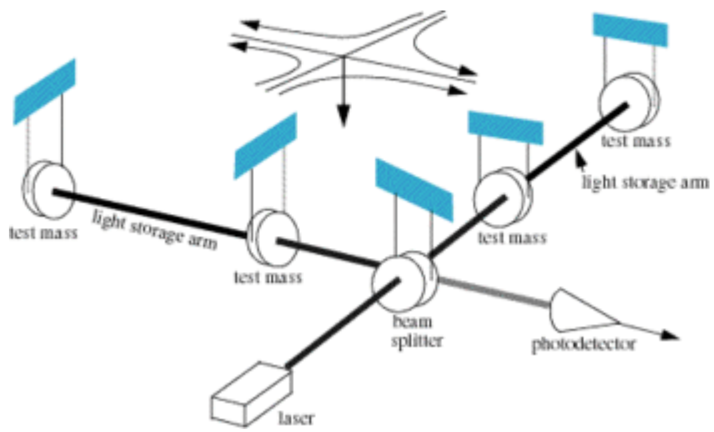
Nobel prize in physics 1993
Russel Hulse & Joseph Taylor

Gravitational wave detectors

- Relative displacement $\frac{\delta L}{L} \sim h$

Typically, $h \sim 10^{-21}$

- Interferometers

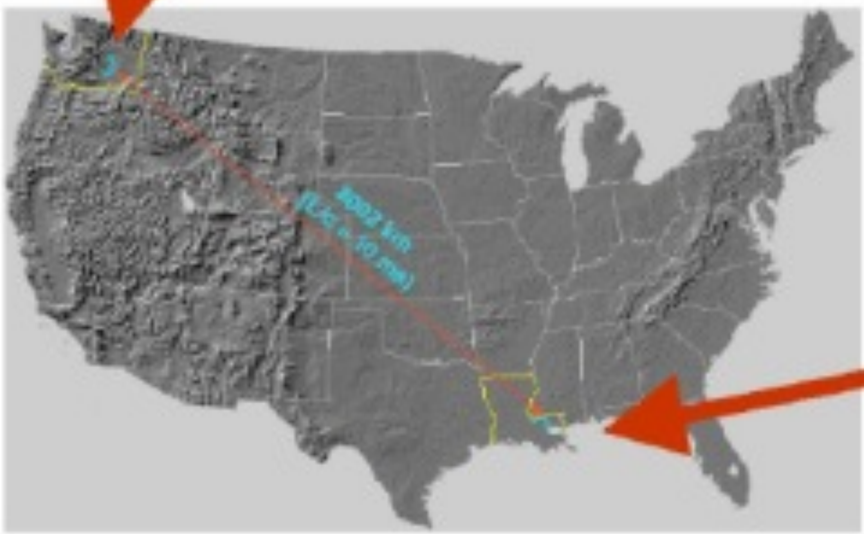


Gravitational wave detectors

Hanford

LIGO

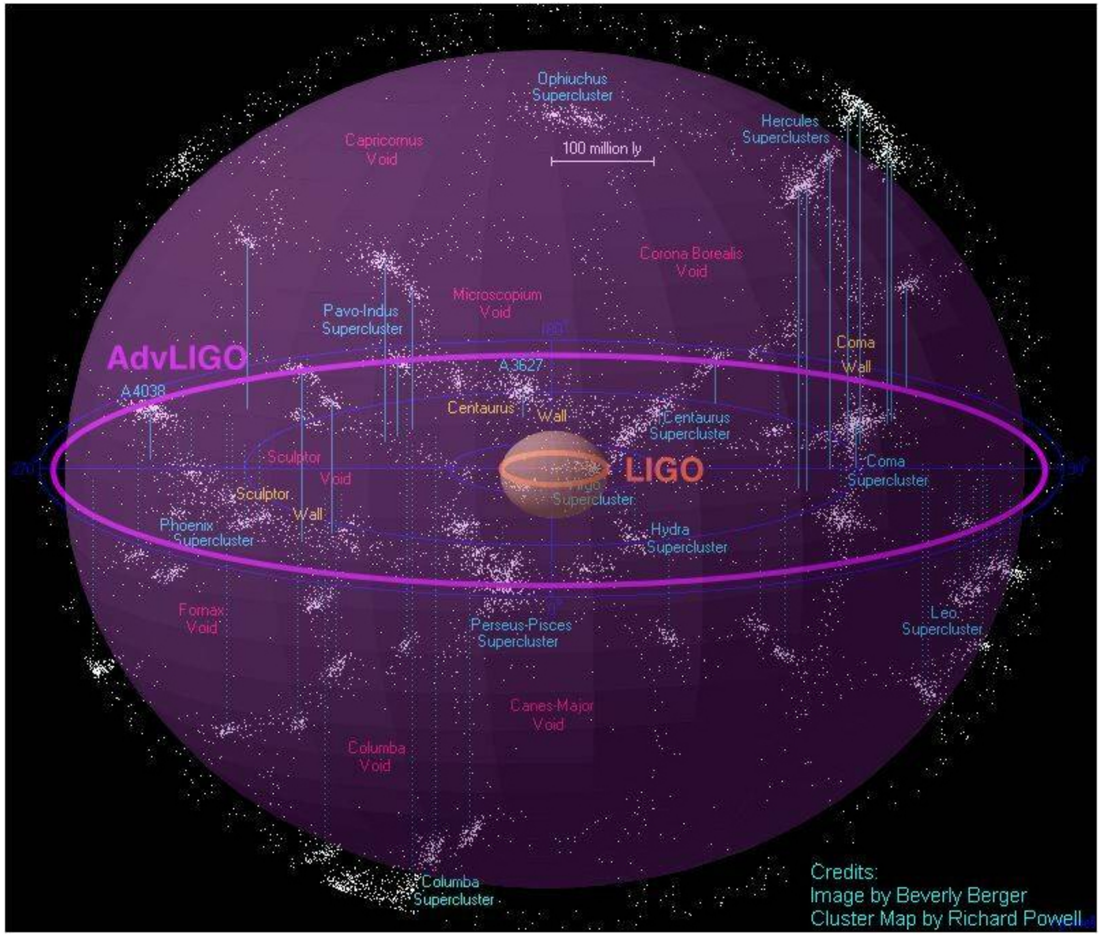
Livingstone



Gravitational wave detectors

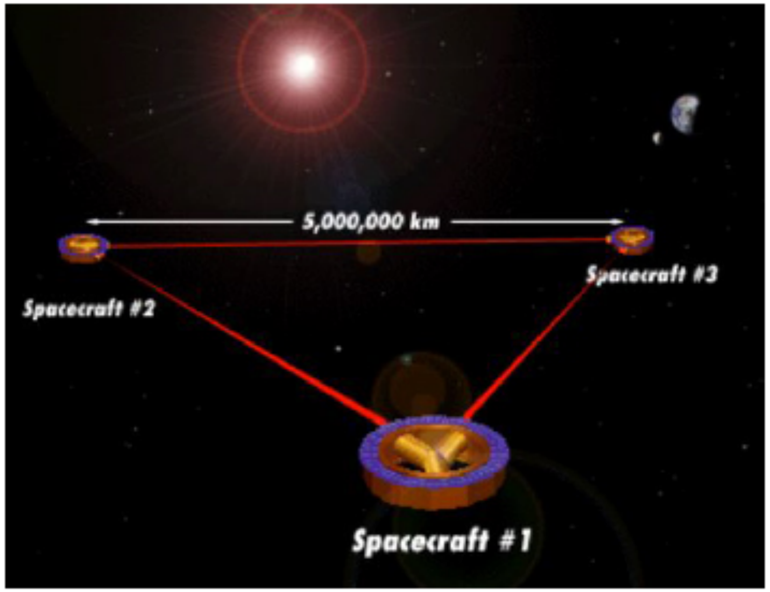
VIRGO, near Pisa

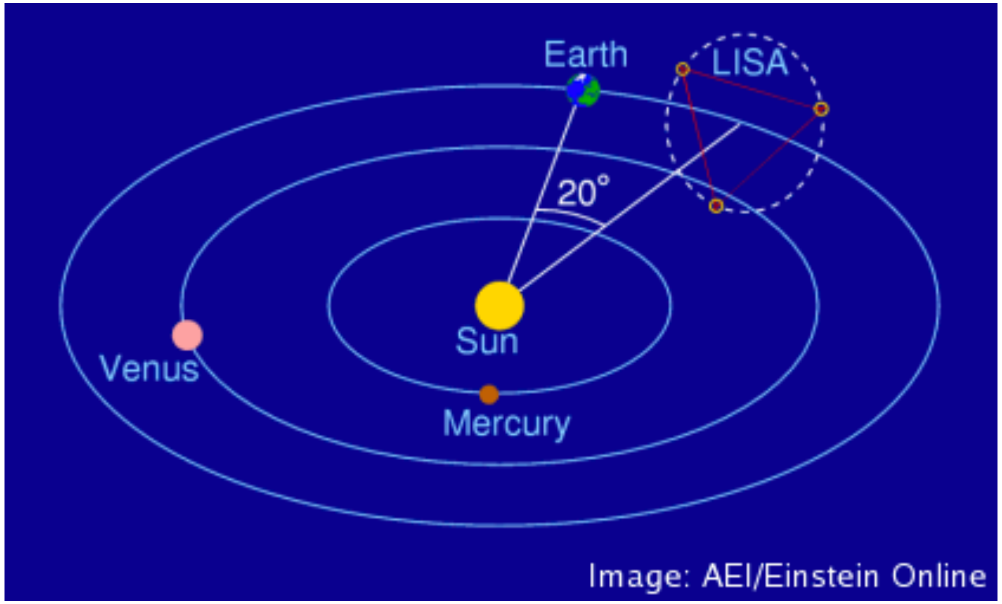
Gravitational wave detectors



Improved sensitivity

Spatial mission eLISA (ESA)





Relativistic cosmology

Relativistic cosmology

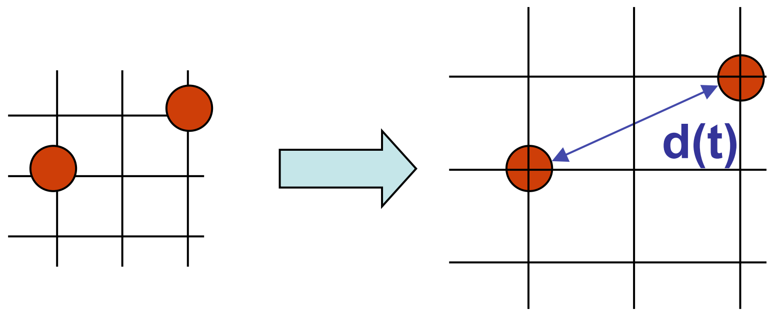
- Symmetries: **spatial homogeneity & isotropy**

- Metric of the form $ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j$

$$\gamma_{ij}\,dx^i dx^j = \begin{cases} d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2) & K > 0 \\ d\chi^2 + \chi^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) & K = 0 \\ d\chi^2 + \mathrm{sh}^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2) & K < 0 \end{cases}$$

$$\gamma_{ij}\,dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \qquad k = -1, 0, 1$$

where $a(t)$ is the scale factor



Friedmann equations

- Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

with the metric $$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$$

and the energy-momentum tensor

$$T^{\nu}_{\mu} = \mathrm{Diag}(-\rho, P, P, P)$$

- This gives Friedmann's equations (1924)

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)\,.$$

Friedmann equations

- Several types of matter: characterized by $w \equiv P/\rho$
  - Non relativistic matter: $P \ll \rho \Rightarrow w = 0$
  - Relativistic matter: $P = \rho/3 \Rightarrow w = 1/3$
- **Evolution of matter**

$$\dot{\rho} + 3H(\rho + P) = 0 \;\Rightarrow\; \rho \propto a^{-3(1+w)} \qquad (\text{const}\, w)$$

- If one species dominates

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \rho \propto a^{-3(1+w)}$$

$$\begin{cases} w = 0: \;\; \rho \propto \dfrac{1}{a^3}, \;\; a(t) \propto t^{2/3} \\ w = \dfrac{1}{3}: \;\; \rho \propto \dfrac{1}{a^4}, \;\; a(t) \propto t^{1/2} \end{cases}$$

$\Rightarrow$ $a(t) \propto t^{\frac{2}{3(1+w)}}$

Cosmological parameters

- Total energy density made of several components

$$\rho = \sum_i \rho_i^{(0)} \left(\frac{a}{a_0}\right)^{-3(1+w_i)} = \sum_i \rho_i^{(0)} (1+z)^{3(1+w_i)}$$

$$\mathcal{H}^2(z) = \frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho - \frac{k}{a^2 H_0^2} = \sum_i \Omega_i (1+z)^{3(1+w_i)},$$

$$\Omega_i \equiv \frac{8\pi G \rho_i^{(0)}}{3H_0^2}, \quad \Omega_k = -\frac{k}{a_0^2 H_0^2}$$

- Example: non-relativistic matter + cosmological constant + k=0

$$\mathcal{H}(z) = \sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)(1+z)^3}$$

Luminosity distance

- Observation of a light source

$$\mathcal{F} = \frac{L_s}{4\pi d_L^2}$$

where $\mathcal{F}$ is the observed flux and L_s the absolute luminosity

$$L_s = \frac{\delta E_s}{\delta t_s} \Rightarrow L_0 = \frac{L_s}{(1+z)^2}$$ ➡ $$d_L = a_0\, r\, (1+z)$$

- In terms of the redshift $(k=0)$

$$-dt^2 + a^2(t)dr^2 = 0 \qquad r_s = \int_{t_s}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0 H_0}\int_0^{z_s} \frac{dz}{\mathcal{H}(z)}$$

$$d_L(z_s) = \frac{1+z_s}{H_0}\int_0^{z_s} \frac{dz}{\mathcal{H}(z)}$$ with $$\mathcal{H}(z) = \mathcal{H}(z;\Omega_i)$$

Supernovae

Supernovae (Ia)

Explosion of a white dwarf that reaches the Chandrasekhar mass

quasi- «standard candle»

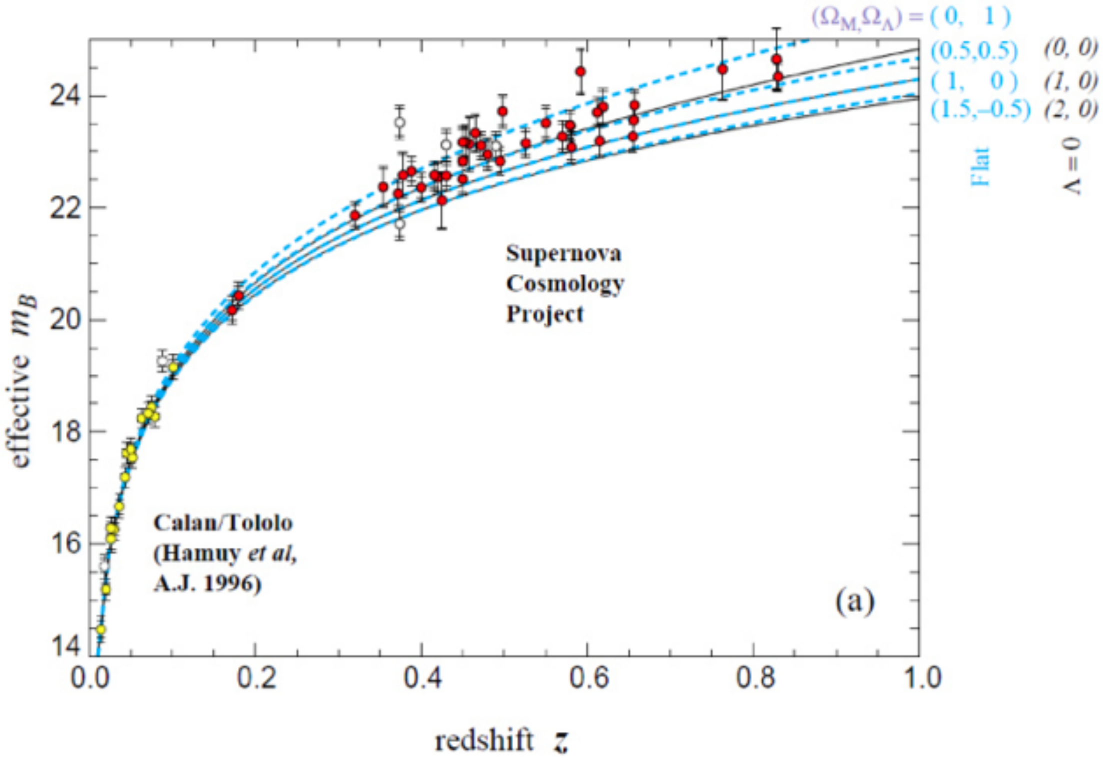
$$m - M = 5\log_{10}\left(\frac{d_L}{\text{Mpc}}\right) + 25$$

m : apparent magnitude

M : absolute magnitude

Accelerated expansion

$$m(z) = M + 5\log_{10}\left(\frac{d_L(z)}{\mathrm{Mpc}}\right) + 25$$



Nobel Prize 2011

« for the discovery of the accelerating expansion of the Universe through observation of distant supernovae »

Conclusions

- General relativity celebrates its 100 years.
- In recent years, it has played a more and more crucial role in astrophysics and cosmology (relativistic stars, stellar and galactic black holes).
- The direct detection of gravitational waves would open a completely new window in astrophysics.
- In cosmology, there have been many attempts to modify general relativity to explain dark energy and dark matter… but observations and internal consistencies are very constraining.