Statistics (3)
Errors

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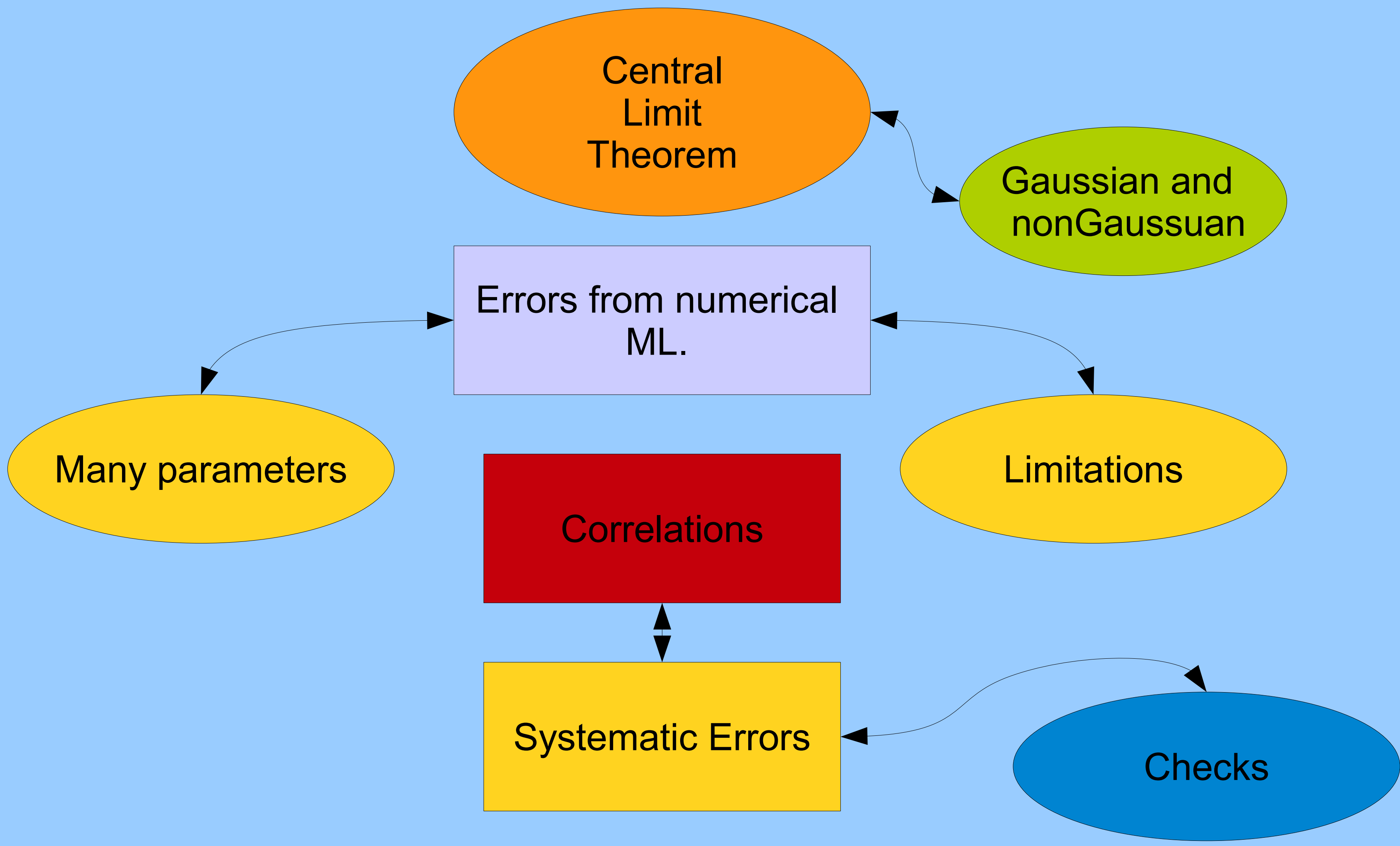
Summary


Central Limit Theorem
Gaussian and nonGaussuan
Errors from numerical ML.
Many parameters
Limitations
Correlations
Systematic Errors
Checks

The Central LimitTheorem

Suppose a random variable x is the sum of several independent identically (or similarly) distributed variables $x_1, x_3, x_3 \ldots x_N$. Then

(1) The mean of the distribution for x is the sum of the means: $\mu = \mu_1 + \mu_2 + \mu_3 + \ldots \mu_N$.

(2) The Variance of the distribution for x is the sum of the Variances: $V = V_1 + V_2 + V_3 + \ldots V_N$.

(3) The distribution for x becomes Gaussian for large N

CLT - why?

Proof is fun but a bit long – see book

Illustration: Take uniform distribution in range 0 to 1..

One on its own is rectangular

Two added give a triangular distribution in range 0 to 2

Three added start with a concanve parabola, switch to a convex one, flat at the top the n back down

Twelve give something so Gaussian it's used to generate random numbers

Thoughts on the CLT parts 1,2,3

(1) Is obvious.

(2) is simple and explains 'adding errors in quadrature.'

(1) and (2) do not depend on the form of the distribution

(3) Explains why Gaussians are 'normal'

If you find a distribution which is not Gaussian, there must be a reason Probably one contribution dominates

Application of CLT

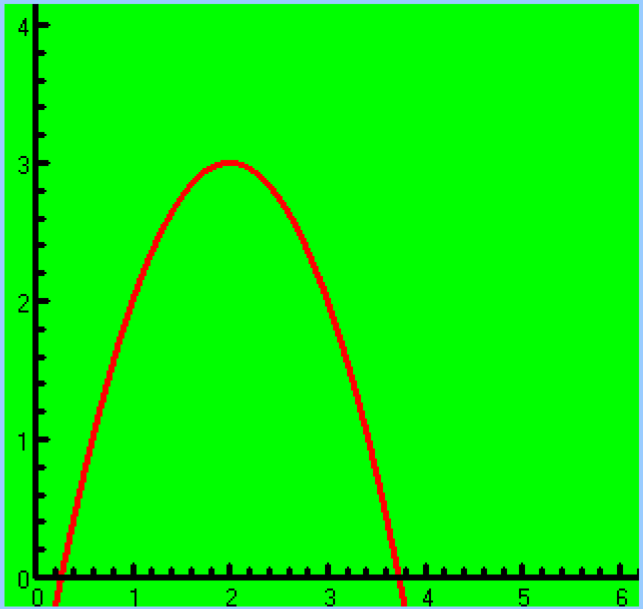
If a variable has a non-Gaussian pdf you can still apply parts (1) and (2): adding variances, using combination of errors, etc.

The only thing you can't do is equate deviations with confidence regions (68% within one sigma etc)

However your variable is probably intermediate and will be a contribution to some final result – Gaussian by (3). So carry on

Non-Gaussian distributions hold no terrors

Errors from Likelihood



Estimate a model parameter M by maximising the likelihood

In the large N limit

i) This is unbiassed and efficient

ii) The error is given by

$$\frac{1}{\sigma^2} = -\left\langle \frac{d^2 \ln L}{dM^2} \right\rangle$$

iii) ln L is a parabola

$$L = L_{max} - \frac{1}{2} C \left(M - \hat{M} \right)^2$$

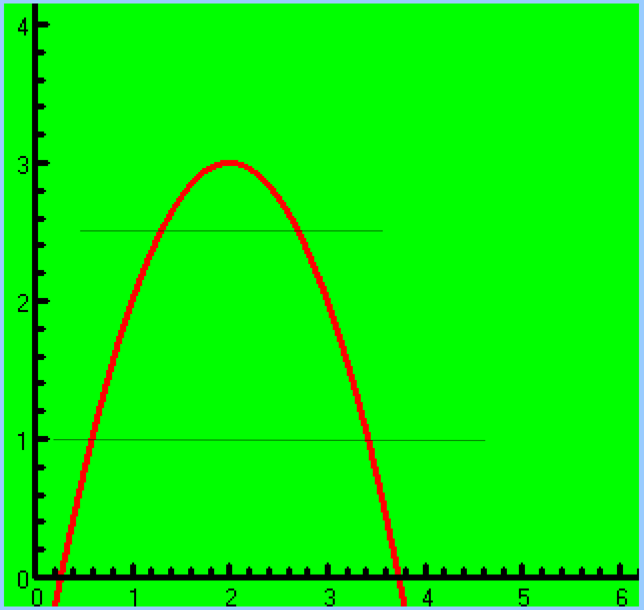
iv) We can approximate

$$C \equiv \frac{-d^2 \ln L}{dM^2}\Big|_{M=\hat{M}} = -\left\langle \frac{d^2 \ln L}{dM^2} \right\rangle$$

v) Read off σ from ΔlnL=-½

Neat way to find Confidence Intervals

- Take $\Delta \ln L = -½$ for 68% CL (1σ)
- $\Delta \ln L = -2$ for 95.4% CL (2σ)
- Or whatever you choose
- 2-sided or 1-sided



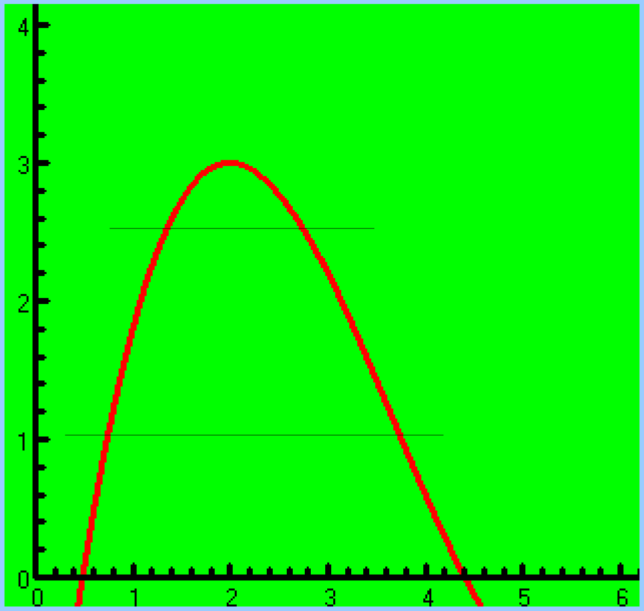
For finite N

None of the above are true

Never mind! We could transform from M $\rightarrow$M' where it was parabolic, find the limits, and transform back

Would give $\Delta \ln L = -\frac{1}{2}$ for 68% CL etc as before

Hence asymmetric errors



Everybody does this

Is it valid?

Try and see with toy model (lifetime measurement) where we can do the Neyman construction

For various numbers of measurements, N, normalised to unit lifetime

There are some quite severe differences!

N	Exact		$\Delta \ln L = -\frac{1}{2}$	
	σ_-	σ_+	σ_-	σ_+
1	0.457	4.787	0.576	2.314
2	0.394	1.824	0.469	1.228
3	0.353	1.194	0.410	0.894
4	0.324	0.918	0.370	0.725
5	0.302	0.760	0.340	0.621
6	0.284	0.657	0.318	0.550
7	0.270	0.584	0.299	0.497
8	0.257	0.529	0.284	0.456
9	0.247	0.486	0.271	0.423
10	0.237	0.451	0.260	0.396
15	0.203	0.343	0.219	0.310
20	0.182	0.285	0.194	0.261
25	0.166	0.248	0.176	0.230

More dimensions

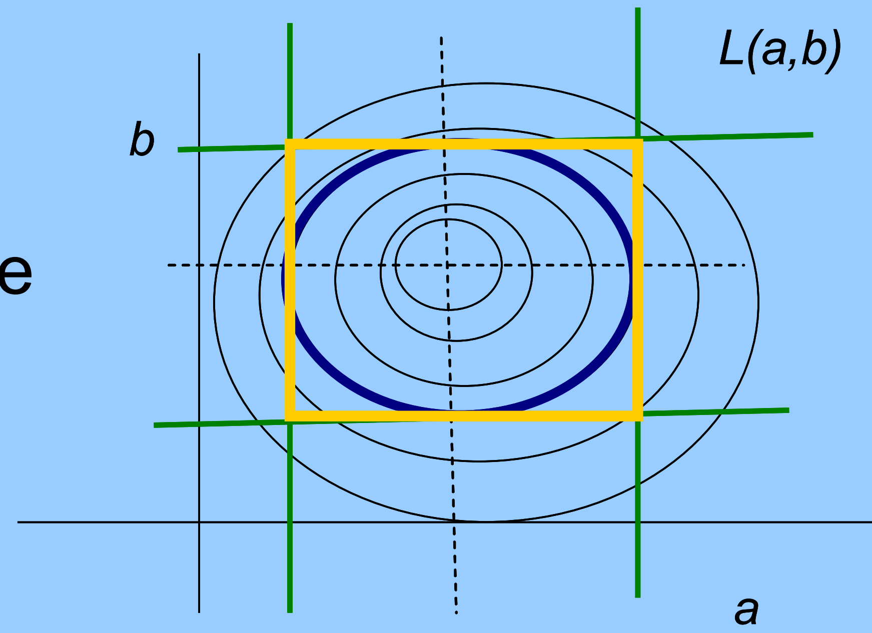
Suppose 2 uncorrelated parameters, a and b

For fixed b, $\Delta \ln L = -\frac{1}{2}$ will give 68% CL region for a

And likewise, fixing a, for b

Confidence level for square is $0.68^2 = 46\%$

Confidence level for ellipse (contour) is 39%



Jointly, $\Delta \ln L = -\frac{1}{2}$ gives 39% CL region

for 68% need $\Delta \ln L = -1.15$

More dimensions, other limits

Useful to write

$$-2\Delta \ln L = \chi^2$$

Careful! Given a multidimensional Gaussian,n $\ln L = -\chi^2/2$. Hence can also use $\Delta\chi^2=1$ for errors

But $-2\Delta \ln L$ obeys a χ^2 distribution only in the large N limit...

Level is given by finding χ^2 such that $P(\chi^2,N)=1-CL$

Generalisation to correlated gaussians is straightforward

Generalisation to more variables is straight forward. Need the larger $\Delta \ln L$

	68%	95%	99%
1	0.5	1.92	3.32
2	1.15	3.00	4.60
3	1.77	3.91	5.65

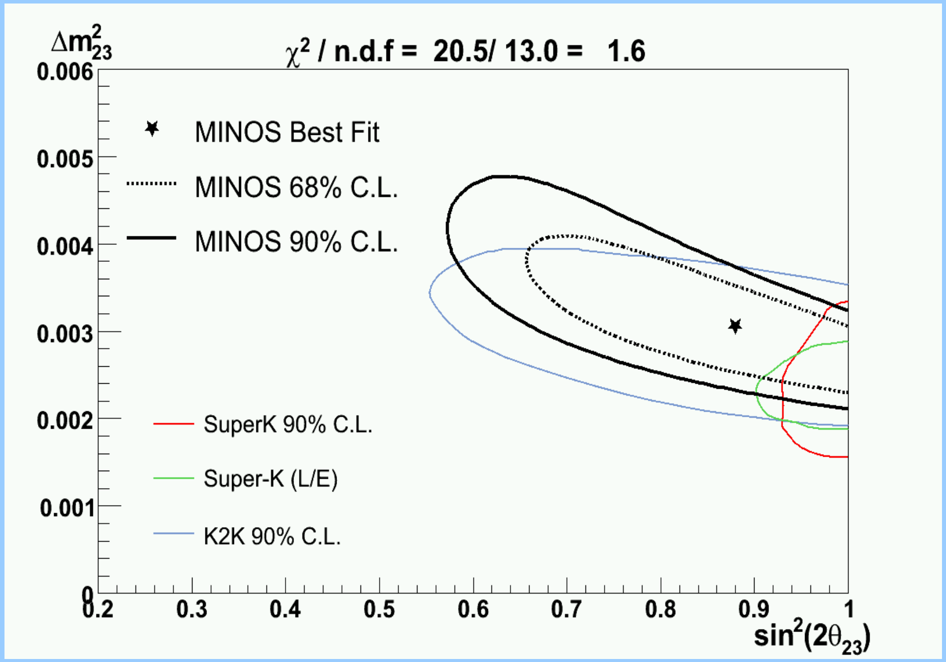
etc

Small N non-Gaussian measurements

No longer ellipses/ellipsoids

Use ΔlnL to define confidence regions, mapping out contours

Probably not totally accurate, but universal



What's the alternative? Toy Monte Carlo

Have dataset

Take point M in parameter space. Is it in or out of the 68% (or ...) contour?

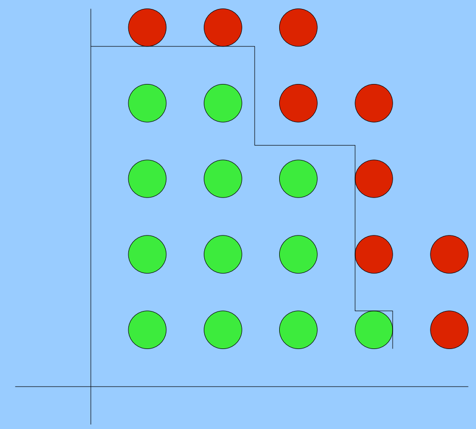
Find $T = \ln L(R|\hat{M}) - \ln L(R|M)$

clearly small T is 'good'

Generate many MC sets of R, using M

How often is $T_{MC} > T_{data}$?

If more than 68%, M is in the contour



We are ordering the points by their value of T (the Likelihood Ratio) – almost contours but not quite

Correlated Errors

Given some f(x,y)

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\rho \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) \sigma_x \sigma_y$$

$$\rho = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\sqrt{(\overline{x^2} - \bar{x}^2)(\overline{y^2} - \bar{y}^2)}}$$

Also matrices

$$V' = \tilde{G} V G$$

Example

Collect N_T events

N_F forwards

N_B backwards

Want error on $R=N_F/N_T$

Everything Poisson

F and B uncorrelated

F and T correlated

$Cov=\langle N_F N_T\rangle - \langle N_F\rangle\langle N_t\rangle$

$=V(N_F)=N_F$

using $\langle N_F N_B\rangle=\langle N_F\rangle\langle N_B\rangle$

$\rho=\sqrt{(N_F/N_T)}$

Continued..

Using R=N_F/N_T

$$\sigma_R^2=\left(\frac{1}{N_T}\right)^2 N_F+\left(\frac{-N_F}{N_T^2}\right)^2 N_T+2\sqrt{N_F/N_T}\left(\frac{1}{N_T}\right)\left(\frac{-N_F}{N_T^2}\right)\sqrt{N_F N_T}$$

$$=\frac{N_F N_T+N_F^2-2N_F^2}{N_T^3}=\frac{R(1-R)}{N_T}$$

Using R=$N_F/(N_F+N_B)$

$$\sigma_R^2=\left(\frac{1}{N_T}-\frac{N_F}{N_T^2}\right)^2 N_F+\left(\frac{-N_F}{N_T^2}\right)^2 N_B=\left(\frac{N_B}{N_T^2}\right)^2 N_F+\left(\frac{N_F}{N_T^2}\right)^2 N_B=\frac{N_F N_B}{N_T^3}=\frac{R(1-R)}{N_T}$$

Systematic Errors

“Systematic errors arise from neglected effects such as incorrectly calibrated equipment.”

Agree or disagree?

Systematic Errors

"Systematic errors arise from neglected effects such as incorrectly calibrated equipment."

FALSE!

A neglected effect is a MISTAKE

A MISTAKE is not an ERROR

(as we tell the undergraduates on day1)

So what are they?

Systematic Errors

Analysis of your results involves a whole set of numerical factors: efficiencies, magnetic fields, dimensions, calibrations...

Occasionally these are implicit: these are especially dangerous

All these numbers have an associated uncertainty.

These uncertainties are the systematic errors. They obey all the usual error laws, but they affect all measurement

Examples

The magnetic field in p= 0.3 B R

Calorimeter energy calibration

'Jet energy scale'

Detector efficiency

...

If you can't think of them for your experiment, ask a colleague with a talent for destructive criticism. (There are plenty around)

How it works...

Effect of uncertainty in B on the error matrix for two momentum measurements

$$V = \begin{pmatrix} 0.3^2 B^2 \sigma_1^2 + 0.3^2 R_1^2 \sigma_B^2 & 0.3^2 R_1 R_2 \sigma_B^2 \\ 0.3^2 R_1 R_2 \sigma_B^2 & 0.3^2 B^2 \sigma_2^2 + 0.3^2 R_2^2 \sigma_B^2 \end{pmatrix}$$

Errors on p_1 and p_2 as given by simple combination of errors.

Also covariance /correlation term. Errors in B effect both momentum measurements the same way

More complicated...

Many properties of the reconstruction don't work through simple algebra.

Example: background to your signal simulated by Monte Carlo containing several (?) adjustable parameters...

Work numerically. Run standard MC, then adjust parameter by +σ and repeat, -σ and repeat   Read off error from shift in result

If you can convince yourself that the 3 points are a straight line then do so

Important guide

Don't sweat the small stuff!

Errors add in quadrature. Go for the biggest. Reducing small errors still further is a waste of your energy:

$\sqrt{(10^2+2^2)}=10.20$

$\sqrt{(10^2+1^2)}=10.05$

Checks

- Check your result by altering features which should make no (significant) difference. This adds to its credibility
- Run on subsets of the data (time etc)
- Change cuts on quality and kinematic quantities
- Check that a full blown analysis on simulated data returns the physics you put in
- Repeat until you (and you supervisor and review committee) really believe

'Significant'

Repeating with some difference in technique will give a different result.

You have to decide whether this is significant.

“Within Errors” may be overgenerous as results share the same data (or some of it)

Subtraction in quadrature is one way:

Basic result 12.3 ± 0.4. Check 11.7 ± 0.5

Compare difference 0.6 against $\sqrt{(.5^2-.4^2)}=.3$

Results of checks

If the analysis passes the check with a small difference

Tick the box and move on

Do not fold that small difference into the systematic error

If the analysis fails the check

1) Check the check
2) Check the analysis and find the problem
3) Maybe convince yourself that this 'harmless' change could cause a systematic shift and devise an appropriate error

Do not fold the difference into the systematic error

Source of confusion

Two tables – similar yet different

Vary

- Energy scale
- Mag field
- Trigger effcy
- MC parameters
- …

and include results in systematic errors

Vary by 1 sigma

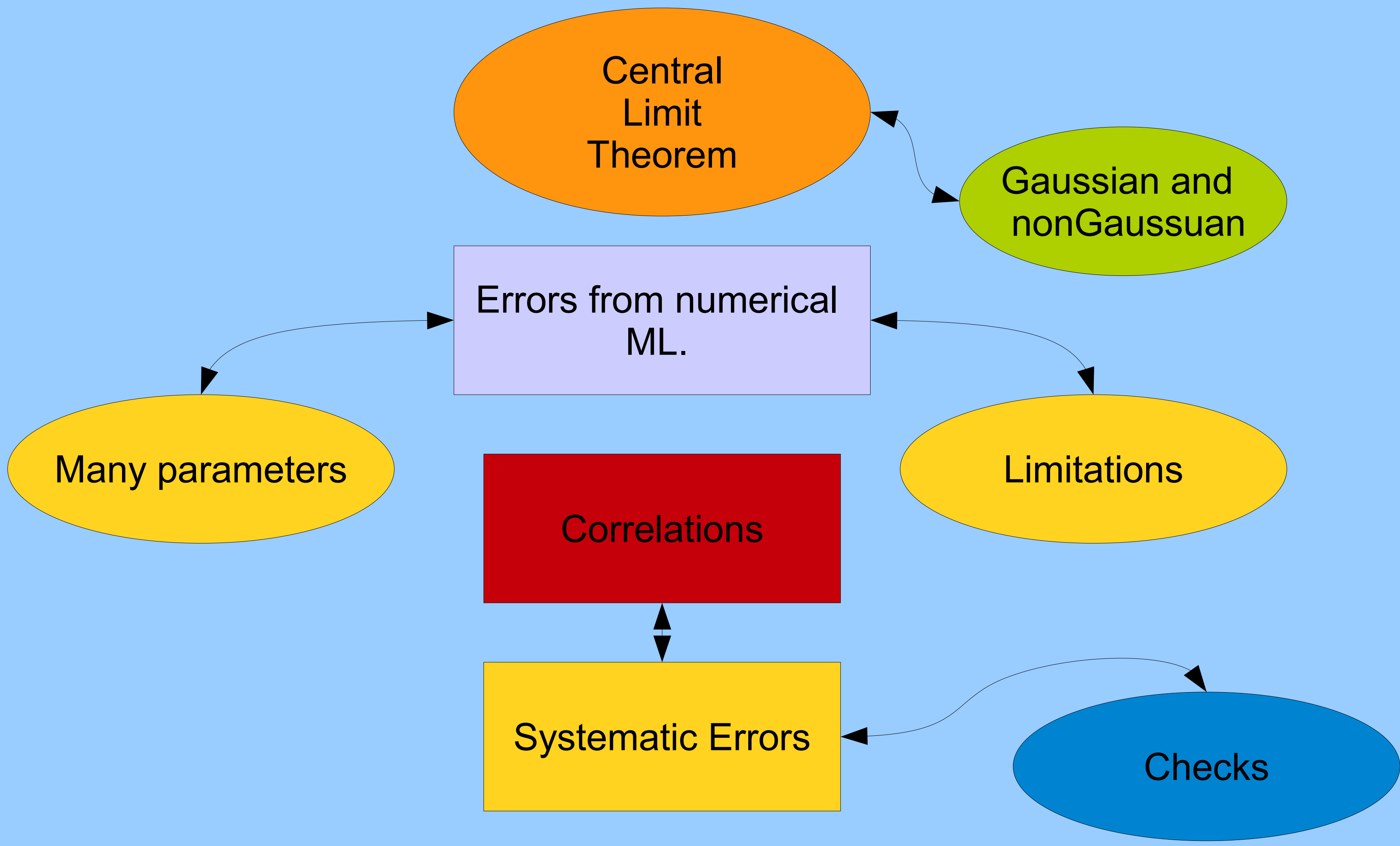
Vary

- Energy cut
- Lepton quality
- isolation
- ...

But do not include results in systematic errors

Vary by Arbitrary amount

Summary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Final thought

Statistics is a science, not an art. There is a reason for everything. Understand what you are doing and why.

Cheap computing is opening many new ways of doing things. Use it!

There is a lot of bad practice out there. Do not take the advice of your supervisor/senior colleague/professor as infallible.